Chapter 1

Polynomial

Definition of polynomial

A polynomial (多項式) is an algebraic expression (代數式) consists of one or more terms (項) where each term is a product of a number and one or more variables (變數) whose exponents (指數) are non-negative integers (非負整數).

Polynomials are simple yet very important tools in mathematics. Examples of polynomials include:

\[ 4a^3, \quad -3x, \quad 2x - 3y, \quad 6x^2 + 7x + 1, \quad 5000x^{5000} + 2x^2 + 9 \]

In the polynomial “ \[ 56xy^3 + 99x - 23y - 33 \] ”:

- \[ 56xy^3 \] , \[ 99x \] , \[ -23y \] and \[ -33 \] are called terms.
- \[ 56, 99, -23 \] and \[ -33 \] are called coefficients (係數)
- \[ x \] and \[ y \] are, of course, variables.
- Since there are 2 variables, this polynomial is called a polynomial with 2 unknowns (二元多項式).
- The term with highest exponent is \[ 56xy^3 \]. The exponent of \[ x \] is 1, and \[ y \] is 3. Thus, the degree (次) or order of this polynomial is 4, and this polynomial is called polynomial of degree 4 (四次多項式).
- The term with highest exponent (\[ 56xy^3 \]) is also called the leading term (領項).
- \[ 99 \] is called the coefficient of \[ x \]. \[ -23 \] is called the coefficient of \[ y \]. Can you guess what is \[ 56 \] called? (Coefficient of \[ xy^3 \])
- \[ -33 \] is called the constant term (常數項), because this term contains no variables. Not all polynomials have constant term.

Functions (函数)

Besides polynomials, also let us introduce the concept of “function” – another useful notation in mathematics – here.

When you calculate the area of a circle with radius \( r \), you will instantly apply formula \( A = \pi r^2 \). Here, \( A \) changes as \( r \) changes. Or we can say the value of \( A \) depends on \( r \). Here, we can call \( A \) is a function of \( r \).

A function is a relation between variables in which the value of one variable, called the dependent variable (應變數) (\( A \)), depends on the value of the other variable(s), called independent variable(s) (獨立變數) (\( r \)). For every value of the independent variable, there is one and only one corresponding value of the dependent variable.

Rather than using \( A \), we often denote a function by \( f(x) \), \( g(y) \), \( F(t) \), \( Q(r) \), etc. These do not mean \( f \) times \( x \), \( g \) times \( y \); but \( f \) is a function of \( x \), \( g \) is a function of \( y \), and so on.

Therefore, the area of circle, \( A \), can be written as:

\[ A(r) = \pi r^2 \]

Assume that we want to find the area of circle with radius 4 cm. In equations, we use \( 4^2 \pi \). In functions, we use the notation \( A(4) \), and the function \( A \) is described above. Here, \( A(4) \) means that “substitute \( r = 4 \)”. Using function can reduce repetition of similar forms in an expression. For example, rather than

\[
\frac{a^2 + a^2 + a + 1 + (b^2 + b^2 + b + 1)(c^2 + c^2 + c + 1) - d^2 - d^2 - d - 1}{17^3 + 17^2 + 17 + 1}
\]

we may just define \( f(x) = x^3 + x^2 + x + 1 \), and the above equation is shortened:

\[
\frac{f(a) + f(b) f(c) - f(d)}{f(17)}
\]
**Addition and subtraction of polynomials**

Addition and subtraction of polynomials can be calculated directly by grouping terms with the same degree and variables. For example:

\[(x^3 + 3x^2 + 3x + 1) + (x^2 + 2x + 1)\]

\[= x^3 + (3x^2 + x^2) + (3x + 2x) + (1 + 1)\]

\[= x^3 + 4x^2 + 5x + 2\]

\[(x^3 + 3x^2 + 3x + 1) - (x^2 + 2x + 1)\]

\[= x^3 + (3x^2 - x^2) + (3x - 2x) + (1 - 1)\]

\[= x^3 + 2x^2 + x\]

\[(x^3 + 3y^2 + 3z + w) + (x^2 + 2y + z)\]

\[= x^3 + 3y^2 + (3z + z) + w + x^2 + 2y\]

\[= x^3 + x^2 + 3y^2 + 2y + 4z + w\]

Notice that we tend to write the polynomials in decreasing order of degrees, also known as descending order (降序).

The result of addition or subtraction of polynomials is always a polynomial.

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**Multiplication of polynomials**

Multiplication of polynomial is, in contrast to addition and subtraction, much more complicated. We can do multiplication by multiplying terms by terms, or in a table form.

*Multiplying terms by terms (逐項乘法)*

This is method is the simplest method. Multiplying terms by terms make use of the fact that \(a(b + c) = ab + ac\).

Take an example:

\[(5x^2 + 6x + 7)(8x^2 - 9x - 10)\]

\[= (5x^2)(8x^2 - 9x - 10) + (6x)(8x^2 - 9x - 10) + (7)(8x^2 - 9x - 10)\]

\[= (40x^4 - 45x^3 - 50x^2) + (48x^3 - 54x^2 - 60x) + (56x^2 - 63x - 70)\]

\[= 40x^4 + 3x^3 - 48x^2 - 123x - 70\]

When a polynomial of degree \(n\) is multiplied by a polynomial of degree \(m\), the result is always a polynomial of degree \(m + n\).

*Multiplying in table form (表格式乘法)*

Besides doing multiplication terms by terms, we can do it differently. Take the above multiplication as example. We noticed that \(8x^2 - 9x - 10\) is multiplied to each term of \(5x^2 + 6x + 7\). If we make a table like this:

<table>
<thead>
<tr>
<th></th>
<th>8x²</th>
<th>-9x</th>
<th>-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x²</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

And multiply the rows and columns together:

<table>
<thead>
<tr>
<th></th>
<th>8x²</th>
<th>-9x</th>
<th>-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>5x²</td>
<td>40x⁴</td>
<td>-45x³</td>
<td>-50x²</td>
</tr>
<tr>
<td>6x</td>
<td>48x³</td>
<td>-54x²</td>
<td>-60x</td>
</tr>
<tr>
<td>7</td>
<td>56x²</td>
<td>-63x</td>
<td>-70</td>
</tr>
</tbody>
</table>

Sum up all the cells, and the resulting polynomial is \(40x^4 + 3x^3 - 48x^2 - 123x - 70\), just the same as terms by terms!

The table can be further simplified to:

<table>
<thead>
<tr>
<th>8</th>
<th>-9</th>
<th>-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>40</td>
<td>-45</td>
</tr>
<tr>
<td>6</td>
<td>48</td>
<td>-54</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>-63</td>
</tr>
</tbody>
</table>

| -70 |
The top left cell (excluding the first row and column), 40, is the coefficient of the leading term. The numbers in the diagonals with direction “/” are coefficients of terms having the same degree. For example, -45 and 48 are coefficients of $x^3$.

This method is best for multiplying polynomial with many terms, and most effective for polynomial with one unknown only.

Note that to achieve the most efficiency, you should arrange the terms in descending order prior to using this method.

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**Useful Polynomial Identities**

An identity (恒等式) is an expression that is always true. For example, $(x+1)+(x+2) = 2x+3$ is an identity, because it is always true regardless what the value $x$ is. We sometimes use the symbol “≡” instead of “=” to show an identity relation, e.g., $(x+1)+(x+2) ≡ 2x+3$.

In this section, we introduce you a few numbers of useful identities. They should be memorized.

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)(a - b) = a^2 - b^2$
4. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

You can prove them by multiplying.

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**Division of polynomial**

Division is much different from, and also harder than, the other 3 operations of polynomial we introduced above.

**Long division (長除法)**

Like doing division in numbers, the long division method also applies for polynomials. The rule is similar, and I in tend not to explain it in detail.

Example:

\[
\begin{array}{c|ccccc}
& x^2 & -8x & +83 \\
\hline
x^2 + 9x -12 & x^4 & +x^3 & -x^2 & -90x & +90 \\
& x^4 & +9x^3 & -12x^2 & \\
\hline
& -8x^3 & +11x^2 & -90x & \\
& -8x^3 & -72x^2 & +96x & \\
\hline
& 83x^2 & -186x & +90 & \\
& 83x^2 & +747x & -996 & \\
\hline
& -933 & +1086 & \\
\end{array}
\]

As expected, $x^2 - 8x + 83$ is called the **quotient** (商), and $-933x + 1086$ is called the **remainder** (餘). A polynomial is said to be **divisible** (整除) by another polynomial if the remainder is zero.

The result of the division is not necessarily a polynomial. Only those are divisible are polynomials.

The relationship between the divisor, dividend, quotient and remainder is illustrated as following:

\[ P(x) = Q(x) F(x) + R(x) \]

Here, $P(x)$ is the dividend $(x^4 + x^3 - x^2 - 90x + 90)$, $Q(x)$ is the quotient, $F(x)$ is the divisor, $R(x)$ is the remainder. Degree of $Q(x)$ is degree of $P(x)$ minus degree of $F(x)$. Degree of $R(x)$ is less than degree of $F(x)$. 

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The long division can also be simplified as follows, by removing the variables:

\[
\begin{array}{c|ccc}
& 1 & -8 & 83 \\
\hline
1 & 9 & -12 & 1 \\
 1 & 9 & -12 & 1 \\
\hline
-8 & 11 & -90 & 90 \\
-8 & 72 & 96 & \\
\hline
83 & -186 & 90 & \\
83 & 747 & -996 & \\
\hline
-933 & 1086 & & \\
\end{array}
\]

**Synthetic Division (對稱除法)**

Synthetic Division is a simpler approach to long division, but only effective for \(F(x) \neq x - a\), where \(a\) is any real numbers (it can be negative!)

For example, take \(P(x) = 7x^3 + 7x^2 - 9x + 1\), \(F(x) = x - 5\).

Write the coefficients of \(P(x)\) on a row:

\[7 \quad 0 \quad 0 \quad 7 \quad -9 \quad 1\]

Put \(a\) (of \(F(x)\)) besides the coefficients of \(P(x)\):

\[
\begin{array}{c|ccc}
5 & 7 & 0 & 0 & 7 & -9 & 1 \\
\hline
7 & 35 & 175 & 875 & 4410 & \\
\hline
7 & 35 & 175 & 882 & 4401 & \\
\end{array}
\]

“Move” the leading coefficient to the bottom:

\[
\begin{array}{c|ccc}
5 & 7 & 0 & 0 & 7 & -9 & 1 \\
\hline
7 & & & & & \\
\hline
7 & & & & & \\
\end{array}
\]

Multiply it by \(a\), put it on the next column in the middle:

\[
\begin{array}{c|ccc}
5 & 7 & 0 & 0 & 7 & -9 & 1 \\
\hline
35 & & & & & \\
\hline
7 & & & & & \\
\end{array}
\]

Add the numbers in that column together and put in the bottom:

\[
\begin{array}{c|ccc}
5 & 7 & 0 & 0 & 7 & -9 & 1 \\
\hline
35 & 175 & & & & \\
7 & 35 & 175 & & & \\
7 & 35 & 175 & 875 & 4410 & \\
7 & 35 & 175 & 882 & 4401 & \\
\hline
22005 & & & & & \\
7 & & & & & \\
7 & & & & & \\
\end{array}
\]

Repeat multiplying, adding…
So what we have got at last? 7, 35, 175, 882, 4401, 22006. What do they mean? Quotient and remainder? Of course, we are doing division here. Actually, all numbers except the last are coefficients of the quotient. The last number is the remainder. So, \( Q(x) = 7x^4 + 35x^3 + 175x^2 + 882x + 4401, \) \( R(x) = 22006. \)

If \( F(x) = mx - n \) rather than \( x - a \), we may take \( a \) as \( \frac{a}{m} \), and divide \( Q(x) \) with the constant \( m \).

For example, when \( P(x) = 7x^3 + 7x^2 - 9x + 1, \) \( F(x) = 2x - 3, \)

\[
\begin{array}{c|cccc}
\frac{1}{2} & 7 & 0 & 0 & 7 \\
2 & 1 & 4 & 198 & 715 & 1773 \\
7 & 21 & 82 & 715 & 1773 \\
2 & 7 & 35 & 245 & 591 & 1805 & 32 \\
7 & 21 & 63 & 591 & 1805 & 32 \\
2 & 7 & 21 & 1805 & 32 \\
\end{array}
\]

\[ Q(x) = \frac{7x^4 + \frac{21}{2}x^3 + \frac{63}{4}x^2 + \frac{245}{8}x + \frac{591}{16}}{2} = \frac{7x^4}{2} + \frac{21x^3}{4} + \frac{63x^2}{8} + \frac{245x}{16} + \frac{591}{32}, \quad R(x) = \frac{1805}{32}. \]

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**Remainder Theorem (餘式定理)**

Sometimes we are only interested in finding the remainder when doing polynomial division. Testing for divisibility is such a case.

The remainder theorem states:

When a polynomial \( P(x) \) is divided by \( x - a \), the remainder \( R \) is \( P(a) \).

Or more generally, when \( P(x) \) is divided by \( mx - n \), the remainder is \( P\left(\frac{n}{m}\right) \).

For example, when \( 9x^3 + 5x^2 - 11x + 56 \) is divided by \( x - 5 \),
the remainder \( = 9(5)^3 + 5(5)^2 - 11(5) + 56 = 1251. \)

The remainder theorem enables us the find out the remainder without doing a complicated division.

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**Revision:**

In this chapter, we’ve learnt:
1. What is polynomial
2. What is function
3. Addition & subtraction of polynomials
4. Multiplication of polynomials
5. Four useful identities
6. Division of polynomials
7. The remainder theorem
Exercise

In the followings, if not specified, \( x \) is the variable, \( k \) is a constant, and \( n \) is a non-negative integer.

1. Show that if the followings is a polynomial or not:
   a) \( x^{2004} + 1 \)
   b) \( x^{2004} - 1 \)
   c) \( x^n + n \)
   d) \( n^3 + x \)
2. Define \( f(x) = x^2 + 6x + 7 \)
   a) Show that \( f(x+y) \neq f(x) + f(y) \)
   b) Show that \( f(xy) \neq f(x)f(y) \)
3. Find the remainder when \( x^{2004} - 1 \) is divided by \( x - 1 \).
4. Show that \( x^{2n} - 1 \) is divisible by \( x + 1 \).
5. Prove remainder theorem.
6. By any method, find out the quotient of \( \frac{x^9 - 1}{x - 1} \)
7. Let \( f(x) = 9x^{1234} - 15x^{5678} + 6 \).
   a) Find out \( [f(x)]^2 \)
   b) Show that \( f(x) \) is divisible by \( x - 1 \).
   c) Show that \( [f(x)]^2 \) is divisible by \( x - 1 \).
8. If two polynomials \( A(x) \) and \( B(x) \) have the same remainder when divided by \( f(x) \), we say that \( A(x) \equiv B(x) \pmod{f(x)} \). By this,
   a) Show that \( A(x) - B(x) \equiv 0 \pmod{f(x)} \).
   b) Show that \( A(x) + a \equiv B(x) + a \pmod{f(x)} \) for any real number \( a \).
   c) If \( x + 8k \equiv x^3 + x^4 + 1 \pmod{x + k} \), find the value of \( k \).
9. (HKMHASC 2003) Evaluate \( \frac{2003125}{2003124 \times 2003126} \)
10. By remainder theorem, prove that if sum of all digits of a number is divisible by 9, then also that number.
    [Hint: 9m - 9n is always divisible by 9 for any integers \( m \) and \( n \)]
Suggested Solutions for the Exercise

1a) Yes  
b) No  
c) Yes  
d) No

3) 0  
6) $x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
7a) $225x^{11356} - 270x^{6912} - 180x^{5678} + 81x^{2468} + 108x^{1234} + 36$
8c) 1/7
9) 2003125