Chapter 2: Linear Equations

Linear Equations

Line in coordinate geometry

In Cartesian coordinate systems, a line can be represented by a linear equation, i.e., a polynomial with degree 1.

But before we proceed, first introduce some terms that should be known.

The figure represents a line. The line crosses the y-axis at \( y = 1 \). The point \((0,1)\) is called the y-intercept (y截距). Similarly, the point \((2,0)\) is called the x-intercept (x截距).

Moreover, there is a quantity called “slope” (斜率) telling us the obliquity of the line. It can be calculated by the following formula:

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Here, \( m \) is the symbol for slope, and \((x_1, y_1), (x_2, y_2)\) are two points lying on the line.

In this line, the slope is \(-1/2\).

If a line is parallel to the x-axis, its slope is 0. If a line is parallel to the y-axis, its slope is undefined.

Equation of a line

The equation of a line can be written as equation

\[Ax + By + C = 0\]

Where \( A, B \) and \( C \) are real constants. This is called the “general form” (一般式) of a line.

We can find the equation of a line by given any two of the following values:

- The slope
- The x-intercept
- The y-intercept
- Coordinate of a point
- Coordinate of another point

Slope-intercept form (斜截式)

When the slope \( m \) and the y-intercept \((0,c)\) of a line is known, then the equation of the line is just:

\[y = mx + c\]

Moreover, base on the above result, we can know the slope and intercept of a line if its general form \( Ax + By + C = 0 \) is given:

\[m = -\frac{A}{B}, \quad c = \frac{C}{B}\]

Intercept form (截距式)

When the x- and y-intercepts, \((a, 0)\) and \((0,b)\) are given, the equation of the line is:

\[\frac{x}{a} + \frac{y}{b} = 1\]

Point-slope form (点斜式)

When the slope \( m \) and a point \((x_0, y_0)\) is given, the equation is:

\[y - y_0 = m(x - x_0)\]
Two-point form (兩點式)

When coordinates of two points \((x_1, y_1), (x_2, y_2)\) are given, the equation is:
\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]

Now take the line above as example. As said, the slope of that line is \(-1/2\), the \(y\)-intercept is \((0,1)\), the \(x\)-intercept is \((2,0)\). We can see that it also passes through the points \((4, -1)\) and \((-2,2)\).

By slope-intercept form, we get the equation:
\[
y = \frac{1}{2}x + 1
\]

By intercept form, we get the equation:
\[
x + y = 1
\]

By point-slope form, we get the equation:
\[
y - y_0 = m(x - x_0)
\]
\[
y - 2 = -\frac{1}{2}(x - (-1))
\]

By two-point form, we get the equation:
\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]
\[
\frac{y + 1}{4 - 2} = \frac{2 + 1}{(-2) - (-1)}
\]

Simplifying all equations above, and we will all get our general formula for the line:
\[
x + 2y - 1 = 0
\]

Intersecting Point of Lines (線之交點)

The following figure shows two lines intersecting at a point:

One may be interested in finding the coordinate of the intersecting point. What should we do? Firstly, find out the equation of the two lines. Here, the two lines are \(2x - y + 3 = 0\) and \(x + y - 1 = 0\) respectively.

Then, we set up a simultaneous equations system (聯立方程系統):
\[
\begin{align*}
2x - y + 3 &= 0 \\
x + y - 1 &= 0
\end{align*}
\]

Since both equations are linear, and there are 2 unknowns, we call this a “simultaneous linear equations in 2 unknowns” (聯立二元一次方程), and I’ll abbreviate it as “SLE2”. For SLE2, we usually write the equations as:
\[
\begin{align*}
2x - y &= -3 \\
x + y &= 1
\end{align*}
\]

Moreover, we would label the first equation as (1), the second as (2):
\[
\begin{align*}
2x - y &= -3 \\
x + y &= 1
\end{align*}
\]

Now what? In order to solve SLE2, we have 3 methods:
**Substitution method (代入法)**

In (2), we see that \( y = 1 - x \). Substituting this into (1), we get:

\[
\begin{align*}
2x - (1-x) &= -3 \\
3x - 1 &= -3 \\
x &= -\frac{2}{3}
\end{align*}
\]

So, \( y = 1 - \left(-\frac{2}{3}\right) = \frac{5}{3} \). That means the intersecting point is \( (\frac{2}{3}, \frac{5}{3}) \).

**Elimination method (消元法)**

Rather than substitution, we may also solve the equation by eliminating a variable. How? In (1), we see a “-x”. In (2), we see a “+y”. Doesn’t it be good that they can be cancelled by adding (1) and (2) together? Sure! You can just add (1) and (2) together to form a new equation:

\[
\begin{align*}
3x &= -2 \\
-3y &= -5
\end{align*}
\]

Not just \( y \), \( x \) can also be eliminated by doing \( 1 - 2 \times (2) \), so:

\[
\begin{align*}
x &= \frac{2}{3} \\
y &= \frac{5}{3}
\end{align*}
\]

**Formula (公式)**

The above two methods are easy to understand and implement, but would be inefficient if the coefficients are quite “complicated”, like, \( \sqrt{2}x + 5y - 8 + \sqrt{7} = 0 \). Actually, there is a general formula solving SLE2. For

\[
\begin{align*}
ax + by &= c \\
dx + ey &= f
\end{align*}
\]

Define \( \Delta = ae - bd, \quad \Delta_x = bf - ce, \quad \Delta_y = cd - af \) (\( \Delta \) is pronounced as “delta”). Then

\[
x = \frac{\Delta_y}{\Delta}, \quad y = \frac{\Delta_x}{\Delta}.
\]

To simplify memorizing, we often denote \( \begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr \), so

\[
\Delta = \begin{vmatrix} a & b \\ d & e \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} b & c \\ e & f \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} c & a \\ f & d \end{vmatrix}.
\]

Note that not all SLE2 are solvable. Sometimes two lines are parallel, i.e., no intersecting point. Using substitution or elimination method will yield a false statement (e.g. 7=0), and using formula will give \( \Delta=0 \), but either \( \Delta_x \) or \( \Delta_y \) is not zero.

On the other hand, an SLE2 may also have infinite solutions. Such a case is two lines overlap. At that time, using substitution or elimination method will give an identity (e.g. 1=1), and using formula will result in \( \Delta, \Delta_x, \) and \( \Delta_y \) all are zeroes.

In addition, substitution method can also apply for other different types of simultaneous equations. For example, consider

\[
\begin{align*}
xy^2 &= 1 \quad \text{(1)} \\
x^{-1} &= 2y \quad \text{(2)}
\end{align*}
\]

By substituting \( x = \frac{1}{2y} \) into (1), we get \( y^2 = 1 \), i.e., \( y = 2 \). And \( x = \frac{1}{4} \) can easily be derived from this.

**Extension:** If more than 2 lines intersect in one point only, they are called “concurrent” (共點).

**Absolute value (絕對值)**

Before we proceed, let’s introduce what is “absolute value”. The absolute value of \( x \) is denoted by \( |x| \), where \( |x| \) is defined as:

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

Or simply: ignore the negative sign. For example, \( |5| = 5; \ |-3| = 3; \ |5-6| = 1 \). The properties of absolute values are:

\[
\begin{align*}
|x| &\geq 0 \\
|x| &= |-x| \\
|xy| &= |x||y| \\
\frac{|x|}{y} &= \frac{|x|}{|y|} \\
|x|^2 &= |x|^2 = x^2
\end{align*}
\]

If \( |x| = a \), and \( a \geq 0 \), then \( x = a \) or \( x = -a \). If \( a < 0 \), the equation has no solutions.

If \( |x| = |y| \), then \( x = y \) or \( x = -y \).
Distance

Let us first review how to calculate the distance between two points. By Pythagoras’ theorem, we know the distance $d$ between two points $(x_1, y_1)$ and $(x_2, y_2)$ is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

How about the distance between a point and a line? First we need to “define” the meaning the distance here, because there are so many points on a line for you to measure. We define the distance between a point and a line to be the shortest distance.

Among all the lines passing though the point and the line, we found that the line that both pass through the point and perpendicular to the line make the distance shortest, as illustrated above.

If the coordinates of a point is $(x_1, y_1)$, and the equation of the line is $ax + by + c = 0$, then the distance is:

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

For example, the graph above tells us that the distance between point $(1, 3/2)$ and line $y = -\frac{x}{2} + 1$ is:

$$d = \frac{|-\frac{1}{2} \times 1 + (-1) \times \frac{3}{2} + 1|}{\sqrt{(-\frac{1}{2})^2 + (-1)^2}} = \frac{2}{\sqrt{5}} \approx 0.894427191...$$

Properties of Slope

Slope tells us how oblique a line is. Actually, by this, we can give out some properties about slope.

1) If the slopes of two lines are the same (i.e., $m_1 = m_2$), they are parallel or the same.
2) If the product of the slopes of two lines is $-1$ (i.e., $m_1m_2 = -1$), they are perpendicular to each other.

Point of Division of a Line (線之分點)

A point of division on a line is the point that divides a line into two parts.

On line $AB$, we define a point $P$ such that it divides the line into ratio of $r: s$. If the coordinates of $A$ is $(x_1, y_1)$, $B$ is $(x_2, y_2)$, $P$ is $(x, y)$, then

$$(x, y) = \left( \frac{sx_1 + rx_2}{s + r}, \frac{sy_1 + ry_2}{s + r} \right)$$

This is called the section formula (分點公式).

Note that $r$ and $s$ may be negative, and the point is said to divide the line externally (外分).

For example, if we want to find the point that divides the line segment $AB$ above in ratio 3:5, by section formula, the point is $\left( \frac{11}{2}, \frac{11}{2} \right)$. 
A special case of section formula is that \( r : s = 1 : 1 \). This yields the **mid-point formula** (中點公式),

\[
(x, y) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

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### Area of Polygons

Last but not least, we tell you how to calculate the area of a polygon with the coordinates of each vertex is given.

If the vertices of a polygon are \((x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\), then its area is:

\[
A = \frac{1}{2} \left| \sum_{i=1}^{n} (x_i, y_i) \right| - \frac{1}{2} \left| \sum_{i=1}^{n} (x_i, y_i) \right|
\]

Here, notation

\[
\begin{vmatrix}
(x_1, y_1) \\
(x_2, y_2) \\
\vdots \\
(x_n, y_n) \\
(x_1, y_1)
\end{vmatrix}
\]

means

\[
(x_1 y_2 + x_2 y_3 + \ldots + x_{n-1} y_n + x_n y_1) - (x_1 y_1 + x_2 y_2 + \ldots + x_{n-1} y_{n-1} + x_n y_n)
\]

Or, we can memorize it like this:

\[
\begin{vmatrix}
-x_2 y_1 \\
x_3 y_2 \\
-x_4 y_3 \\
\vdots \\
x_n y_1
\end{vmatrix}
\]

Note that the points \((x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\) should be arranged in anti-clockwise or clockwise order. Otherwise, the result may be wrong.

For example, the area of the following quadrilateral is:

\[
A = \frac{1}{2} \begin{vmatrix}
3 & -1 \\
2 & \frac{1}{2} \\
2 & -\frac{1}{2} \\
3 & -1
\end{vmatrix} = \frac{1}{2} \begin{vmatrix} 3 \end{vmatrix} + \begin{vmatrix} \frac{3}{2} \end{vmatrix} = \frac{3}{2}
\]

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**Revision:**

In this chapter, we’ve learnt:

1. Line in coordinate geometry
2. Equation of a line
3. Intersecting point of lines
4. Absolute value
5. Distance between line and point
6. Properties of slope
7. Point of division
8. Area of polygon
Exercise

In the followings, if not specified, \( k \) is a constant, and \( n \) is an integer.

1. Find out the slope, \( x \)- and \( y \)-intercept of the followings:
   a) \( 5x + 6y + 7 = 0 \)
   b) \( y = x - 9 \)
   c) \( \frac{x}{4} + \frac{y}{3} = 2 \)
   d) \( x = y \)

2. Find out the equation of a line which the product of its \( x \)- and \( y \)-intercept is 20, the \( x \)-intercept is on the right of the origin, and it is parallel to the line \( 5x + 2y - 6 = 0 \).

3. (HKCEE 1990) In the figure, \( A \) (3,0), \( B \) (0,5) and \( C \) (0,1) are three points and \( O \) (0,0) is the origin. \( D \) is a point on \( AB \) such that the area of \( \Delta BCD \) equals half of the area of \( \Delta OAB \). Find the equation of the line \( CD \).

4. In line segment \( AB \), the coordinates of the two ends \( A \) and \( B \) are (2, 5) and (7, 3) respectively. \( C \) (\( k, k \)) and \( P \) (\( x_0, y_0 \)) is a point on \( AB \). \( D \) (3, 4) is a point. Find:
   a) The equation of \( AB \).
   b) \( k \).
   c) The ratio \( r:s \) that point \( C \) divides \( AB \). [Hint: \( r:s = r/s \)]
   d) Length of \( CD \).
   e) The shortest length of \( PD \).
   f) The coordinates of \( P \) when \( PD \) is the shortest.
   g) Length of \( CP \) when \( PD \) is the shortest.
   h) Area of \( \Delta PCD \) when \( PC \) is the shortest.

5. Prove the formula for solving an SLE.

6. (HKMO 2000 Heat) Find the shortest distance between the line \( 3x - y - 4 = 0 \) and the point (2, 2).

7. If \(|x - 3| + |x - 5| = 4\), find all possible values of \( x \).
   [Hint: Consider the conditions for \( x < 3 \), \( 3 \leq x < 5 \) and \( x \geq 5 \)].

8. (ISM 2000 Final) Solve \( x, y \), and \( z \) in:
   \[
   \begin{align*}
   x - y + z &= 1 \\
   \frac{x}{8} - \frac{y}{4} + \frac{z}{2} &= 1 \\
   \frac{x}{27} - \frac{y}{9} + \frac{z}{3} &= 1
   \end{align*}
   \]

9. Consider the lines \( L_1: x - y + 4 = 0 \) and \( L_2: 3x + y - 12 = 0 \). Find the equation of the line passing through the intersection point of \( L_1 \) and \( L_2 \), and is perpendicular to \( L_1 \).

10. A line \( L \) intersects the axes at \( A \) (\( a, 0 \)) and \( B \) (0, \( b \)). \( M \) (-3, 4) is the mid-point of \( AB \).
    a) Find \( a \) and \( b \).
    b) Find the equation of \( AB \).
    c) \( C \) is a point on the coordinate plant such that \( AC = BC \). The area of \( \Delta BCD \) is 25 square units. Find all possible coordinates of \( C \).

11. Prove \(|xy| = |x| \cdot |y| \).

12. Let \( A \) (\( x_1, y_1 \)), \( B \) (\( x_2, y_2 \)) and \( C \) (\( x_3, y_3 \)) be points of vertices of a triangle. Let \( D, E, F \) be midpoints of \( BC, CA \) and \( AB \) respectively.
    a) Show that \( AD, BE \) and \( CF \) are concurrent.
    b) Assume the three lines meet at \( G \). Show that \( AG:GD = BG:GE = CG:GF = 2:1 \)
Suggested Solutions for the Exercise

1a) Slope = -5/6, y-intercept = -7/6, x-intercept = -7/5
   b) Slope = 1, y-intercept = -9, x-intercept = 9
   c) Slope = -3/4, y-intercept = 6, x-intercept = 8
   d) Slope = 1, y-intercept = 0, x-intercept = 0

2) $5x + 2y - 10\sqrt{2} = 0$
3) $7x - 15y + 15 = 0$

4a) $2x + 5y - 29 = 0$
   b) $\frac{29}{7}$
   c) $\frac{3}{4}$
   d) $\frac{\sqrt{65}}{7}$
   e) $\frac{3}{\sqrt{29}}$
   f) $\left(\begin{array}{c}93 \\ 29 \\ 29 \end{array}\right)$
   g) $\frac{38}{7\sqrt{29}}$
   h) $\frac{57}{203}$

6) 0
7) 2, 6
8) $x=6, y=11, z=6$
9) $x + y - 8 = 0$

10a) $a = -6, b = 8$
    b) $4x - 3y + 24 = 0$
    c) (-7,7), (1,1)