**Formulae for Reference**

\[
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B
\]

\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B
\]

\[
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\]

\[
2 \sin A \cos B = \sin(A + B) + \sin(A - B)
\]

\[
2 \cos A \cos B = \cos(A + B) + \cos(A - B)
\]

\[
2 \sin A \sin B = \cos(A - B) - \cos(A + B)
\]

\[
\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}
\]

\[
\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}
\]

\[
\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}
\]

\[
\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}
\]
**Section A**

This part contains 62 marks.

1) Find:

a) \( \int \cos(3x + 1) \, dx \)

b) \( \int (2 - x)^{2004} \, dx \)

(4 marks)

**Answer:**

a) \( \int \cos(3x + 1) \, dx = \frac{1}{3} \sin (3x + 1) + c \)

b)

\[
\begin{align*}
\int (2 - x)^{2004} \, dx &= \int (x - 2)^{2004} \, dx \\
&= \frac{(x - 2)^{2005}}{2005} + c
\end{align*}
\]

2a) Expand \((1 + 2x)^6\) in ascending powers of \(x\) up to the term \(x^3\).

b) Find the constant term in the expansion of \( (1 - \frac{1}{x} + \frac{1}{x^2}) (1 + 2x)^6 \).

(4 marks)

**Answer:**

a)

\[
(1 + 2x)^6 = 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + \ldots
\]

\[= 1 + 12x + 60x^2 + 120x^3 + \ldots
\]

b)

\[
(1 - \frac{1}{x} + \frac{1}{x^2})(1 + 2x)^6
\]

\[= \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)(1 + 12x + 60x^2 + 120x^3 + \ldots)
\]

\[= 1 - 12 + 60 + \ldots
\]

\[= 49 + \ldots
\]

\[\therefore\] The constant term is 49.
3) The slope at any point \((x, y)\) of a curve \(C\) is given by \(\frac{dy}{dx} = 3x^2 + 1\). If the \(x\)-intercept of \(C\) is 1, find the equation of \(C\). (4 marks)

**Answer:**

Let the curve \(C\) be \(y = y(x)\).

Since the \(x\)-intercept of \(C\) is 1, we have \(y(0) = 1\).

\[ \therefore \frac{dy}{dx} = 3x^2 + 1 \]

\[ \therefore y(x) = \int \frac{dy}{dx} \, dx \]

\[ = \int 3x^2 + 1 \, dx \]

\[ = x^3 + x + c \]

But \(y(0) = 1\)

\[ \therefore c = 1 \]

\[ \therefore y(x) = x^3 + x + 1 \]

\(\therefore\) The equation of \(C\) is \(y = x^3 + x + 1\).

4)

In Figure 1, the shaded region is bounded by the circle \(x^2 + y^2 = 9\), the \(x\)-axis, the \(y\)-axis and the line \(y = 2\). Find the volume of the solid generated by revolving the region about the \(y\)-axis. (4 marks)
The equation of the circle in quadrant I is \( x = \sqrt{9 - y^2} \)

\[ \therefore \text{ The volume of the solid} \]

\[ = \int_0^2 \pi \left( \sqrt{9 - y^2} \right)^2 \, dy \]

\[ = \pi \int_0^2 (9 - y^2) \, dy \]

\[ = \pi \left( 9 - \frac{4}{3} \right) \]

\[ = 4\pi \]

---

5) Find the general solution of the equation

\[ \sin 3x + \sin x = \cos x. \]

**Answer:**

\[ \sin 3x + \sin x = \cos x \]

\[ 2 \sin 2x \cos x = \cos x \]

\[ \sin 4x = \cos x \]

\[ \cos x = \cos \left( \frac{\pi}{2} - 4x \right) \]

\[ \therefore \quad x = 2n\pi \pm \left( \frac{\pi}{2} - 4x \right) \quad \text{for some integer } n \]

\[ x = 2n\pi \pm \frac{\pi}{3} \mp 4x \]

\[ \therefore \quad 5x = 2n\pi + \frac{\pi}{3} \quad \text{or} \quad -3x = 2n\pi - \frac{\pi}{3} \]

\[ \therefore \quad x = \frac{\pi}{15} + \frac{\pi}{3} n\pi \quad \text{or} \quad x = \frac{\pi}{6} - \frac{\pi}{3} n\pi \]

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6)

In Figure 2, \( OAB \) is a triangle. \( C \) is a point on \( AB \) such that \( AC : CB = 1 : 2 \). Let \( \overrightarrow{OA} = \mathbf{a} \) and \( \overrightarrow{OB} = \mathbf{b} \).

a) Express \( \overrightarrow{OC} \) in terms of \( \mathbf{a} \) and \( \mathbf{b} \).
b) If \(|a| = 1, |b| = 2\) and \(\angle AOB = \frac{2\pi}{3}\), find \(|\overrightarrow{OC}|\).

**Answer:**

a) \(
\overrightarrow{OC} = \frac{b + 2a}{1 + 2} = \frac{2}{3}a + \frac{1}{3}b
\)

b)

\[
|\overrightarrow{OC}| = \sqrt{\overrightarrow{OC} \cdot \overrightarrow{OC}}
\]

\[
= \sqrt{\left(\frac{2}{3}a + \frac{1}{3}b\right) \cdot \left(\frac{2}{3}a + \frac{1}{3}b\right)}
\]

\[
= \sqrt{\frac{4}{9}|a|^2 + \frac{1}{9}|b|^2 + \frac{2}{9}a \cdot b}
\]

\[
= \sqrt{\frac{4}{9} + \frac{4}{9}|a||b|\cos \frac{2\pi}{3}}
\]

\[
= \sqrt{\frac{4}{9} + \frac{4}{9}(2)(-\frac{1}{2})}
\]

\[
= \sqrt{\frac{4}{9}}
\]

\[
= \frac{2}{3}
\]

7) Prove that \(9^n - 1\) is divisible by 9 for all positive integers \(n\).

**Answer:**

Let \(S(n)\) be the statement that “\(9^n - 1\) is divisible by 8”.

When \(n = 1\),

\(9^1 - 1 = 8\), which is divisible by 8

\(\therefore S(1)\) is true

Assume \(S(k)\) is true, i.e., \(9^k - 1 = 8m\) for some integer \(m\)

When \(n = k + 1\),

\(9^{k+1} - 1 = 9(9^k - 1) + 9 - 1\)

\[= 9(8m) + 8\]

\[= 8(9m + 1), \text{ which is divisible by 8}\]

\(\therefore S(k + 1)\) is true

\(\therefore S(n)\) is true for all positive integers \(n\).

8) Solve the following equations:

a) \(|x - 3| = 1\)

b) \(|x - 1| = |x^2 - 4x + 3|\)

(6 marks)
Answer:

a) 
\[ |x - 3| = 1 \]
\[ \therefore \quad x - 3 = 1 \quad \text{or} \quad x - 3 = -1 \]
\[ \therefore \quad x = 4 \text{ or } 2 \]

b) 
\[ |x - 1| = |x^2 - 4x + 3| \]
\[ |x - 1| = |x - 1||x - 3| \]
\[ 0 = |x - 1||x - 3| - 1 \]
\[ \therefore \quad |x - 1| = 0 \quad \text{or} \quad |x - 3| = 1 \]
\[ \therefore \quad x = 1 \quad \text{or} \quad x = 2 \text{ or } 4 \quad \text{(by (a))} \]
\[ \therefore \quad x = 1, 2 \text{ or } 4 \]

9)

In Figure 3, \( P(a, b) \) is a point of the curve \( C: y = x^3 \). The tangent to \( C \) at \( P \) passes through the point \((0, 2)\).

a) Show that \( b = 3a^3 + 2 \)

a) Find the values of \( a \) and \( b \).

(6 marks)
Answer:

a) 
Let the tangent be \( y = mx + 2 \)
\[ m = y'(a) \]
\[ = 3a^2 \]
\[ \therefore b = (3a^2)(a) + 2 \]
\[ = 3a^3 + 2 \]

b) 
\[ \because P \text{ is on } C \]
\[ \therefore b = a^3 \]
But we also have \( b = 3a^3 + 2 \)
\[ \therefore a^3 = 3a^3 + 2 \]
\[ -2a^3 = 2 \]
\[ a = -1 \]
\[ \therefore b = (1)^3 = -1 \]

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10) Let \( O \) be the origin and \( A \) be the point \((3, 4)\). \( P \) is a variable point such that the area of \( \Delta OPA \) is always equal to 2.
Show that the locus of \( P \) is a pair of parallel lines.
Find the distance between these two lines.  

(6 marks)

Answer:

Let the coordinates of \( P \) be \((x, y)\)
\[ \therefore \frac{1}{2} \begin{vmatrix} 0 & 0 \\ 3 & 4 \\ x & y \\ 0 & 0 \end{vmatrix} = 2 \]
\[ \begin{vmatrix} 3y - 4x \end{vmatrix} = 4 \]
\[ \therefore 3y - 4x = \pm 4 \]
\[ \therefore 4x - 3y \pm 4 = 0 \] is the locus of \( P \), which is a pair of parallel lines.

The distance between these two lines
\[ = \frac{4 - (-4)}{\sqrt{4^2 + (-3)^2}} \]
\[ = \frac{8}{5} \]
In Figure 4, $OABC$ is a pyramid such that $OA = 3$, $OB = 5$, $BC = 12$, $\angle AOC = 120^\circ$ and $\angle OAB = \angle OBC = 90^\circ$.

a) Find $AC$.

b) A student says that the angle between planes $OBC$ and $ABC$ can be represented by $\angle OBA$.

Determine whether the student is correct or not.

(6 marks)

**Answer:**

a)

In $\triangle OBC$,

$$OC = \sqrt{5^2 + 12^2} = 13$$

In $\triangle OAC$,

$$AC = \sqrt{3^2 + 13^2 - 2(3)(13)\cos 120^\circ} \quad \text{(cosine law)}$$

$$= \sqrt{178 + 39}$$

$$= \sqrt{217} \approx 14.7$$

b)

No.

The angle between the two planes can be represented by $\angle OBA$ if and only if $\angle ABC$ is a right angle.

But since:

$$AB^2 + BC^2 = (5^2 - 3^2) + 12^2$$

$$= 160$$

$$\neq AC^2$$
Therefore, by converse of Pythagoras’ theorem, $\angle ABC$ is not right. This means the angle between $OBC$ and $ABC$ cannot be represented by $\angle OBA$.

12)

Figure 5 shows two lines $L_1$: $y = -x + c$ and $L_2$: $y = 2x$, where $c > 0$. The two lines intersect at point $P$.

a) Let $\theta$ be the acute angle between $L_1$ and $L_2$. Find $\tan \theta$.

b) $L_1$ intersects the $x$- and $y$-axes at points $A$ and $B$ respectively. Find $AP : PB$.

(7 marks)

Answer:

a) $\tan \theta = \frac{2 - (-1)}{1 + (-1)(2)} = \frac{3}{-1} = -3$

b)
Coordinates of $B = (c, 0)$
Coordinates of $A = (0, c)$

Sub $L_2$ into $L_1$:

$$2x = -x + c$$

$$x = \frac{c}{3}$$

$\therefore L_2$:

$$y = 2x = \frac{2c}{3}$$

$\therefore$ Coordinates of $P = \left( \frac{c}{3}, \frac{2c}{3} \right)$

$\therefore$

$$\frac{AP}{PB} = \frac{\sqrt{(c - \frac{2c}{3})^2 + \left(\frac{2c}{3}\right)^2}}{\sqrt{(c - \frac{c}{3})^2 + \left(\frac{2c}{3}\right)^2}}$$

$$= \frac{2\left(\frac{c}{3}\right)^2}{2\left(\frac{c}{3}\right)^2}$$

$$= \frac{1}{4}$$

$\therefore AP : PB = 1 : 4$
Section B
This part contains 48 marks.
Only four questions are needed to be answered.

13) In Figure 6, OABC and ODEF are two squares such that OA = 1, OF = 2 and ∠COD = θ, where 0° < θ < 90°. Let \( \overrightarrow{OD} = 2\mathbf{i} \) and \( \overrightarrow{OF} = -2\mathbf{j} \), where \( \mathbf{i} \) and \( \mathbf{j} \) are two perpendicular unit vectors.

   a) i) Express \( \overrightarrow{OC} \) and \( \overrightarrow{OA} \) in terms of \( \theta \), \( \mathbf{i} \) and \( \mathbf{j} \).

   ii) Show that \( \overrightarrow{AD} = (2 + \sin \theta)\mathbf{i} - \cos \theta\mathbf{j} \)  

b) Show that \( \overrightarrow{AD} \) is always perpendicular to \( \overrightarrow{FC} \).  

   c) Find the value(s) of \( \theta \) such that the points B, C and E are collinear. Give your answer(s) correct to the nearest degree.
Answer:

ai) Let \( \overrightarrow{OC} = ai + bj \), where \( a, b > 0 \).
\[
\begin{align*}
    a^2 + b^2 &= 1 \quad \text{...(1)} \\
    \frac{b}{a} &= \tan \theta \quad \text{...(2)}
\end{align*}
\]
Sub (2) into (1):
\[
\begin{align*}
    a^2 + a^2 \tan^2 \theta &= 1 \\
    a^2 \sec^2 \theta &= 1 \\
    a &= \cos \theta
\end{align*}
\]
∴ (2):
\[
    b = \sin \theta
\]
∴ \( \overrightarrow{OC} = \cos \theta \hat{i} + \sin \theta \hat{j} \)

Let \( \overrightarrow{OA} = ci + dj \), where \( c < 0 \) and \( d > 0 \).
\[
\begin{align*}
    c^2 + d^2 &= 1 \quad \text{...(3)} \\
    \frac{d}{c} &= \tan(90^\circ + \theta) \quad \text{...(4)}
\end{align*}
\]
Sub (4) into (3):
\[
\begin{align*}
    c^2 + c^2 \cot^2 \theta &= 1 \\
    c^2 \csc^2 \theta &= 1 \\
    c &= -\sin \theta
\end{align*}
\]
∴ (4):
\[
    d = -\sin \theta (-\cot \theta) = \cos \theta
\]
∴ \( \overrightarrow{OA} = -\sin \theta \hat{i} + \cos \theta \hat{j} \)

ii)
\[
\begin{align*}
    \overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA} \\
    &= 2\hat{i} - (\sin \theta \hat{i} - \cos \theta \hat{j}) \\
    &= (2 + \sin \theta) \hat{i} - \cos \theta \hat{j}
\end{align*}
\]
b) 
\[ \overrightarrow{FC} = \overrightarrow{OC} - \overrightarrow{OF} \]
\[ = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + 2 \mathbf{j} \]
\[ = \cos \theta \mathbf{i} + (2 + \sin \theta) \mathbf{j} \]
\[ \therefore \overrightarrow{AD} \cdot \overrightarrow{FC} = \left( (2 + \sin \theta) \mathbf{i} - \cos \theta \mathbf{j} \right) \cdot \left( \cos \theta \mathbf{i} + (2 + \sin \theta) \mathbf{j} \right) \]
\[ = (2 + \sin \theta) \cos \theta - \cos \theta (2 + \sin \theta) \]
\[ = 0 \]
\[ \therefore \overrightarrow{AD} \perp \overrightarrow{FC} \]

e) 
\[ \overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC} \]
\[ = 2 \mathbf{i} - 2 \mathbf{j} - \cos \theta \mathbf{i} - \sin \theta \mathbf{j} \]
\[ = (2 - \cos \theta) \mathbf{i} - (2 + \sin \theta) \mathbf{j} \]
\[ \overrightarrow{CB} = \overrightarrow{OA} \]
\[ = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \]
\[ \therefore B, C, E \text{ are collinear} \]
\[ \therefore \overrightarrow{CE} = k \overrightarrow{CB} \text{ for some constant } k \]
\[ \begin{cases} 
2 - \cos \theta = -k \sin \theta \\ 
2 + \sin \theta = -k \cos \theta 
\end{cases} \quad ...(1) \]
\[ \begin{cases} 
2 - \cos \theta = -k \sin \theta \\ 
2 + \sin \theta = -k \cos \theta 
\end{cases} \quad ...(2) \]
\[ \frac{(1)}{(2)}:\]
\[ \frac{\sin \theta}{\cos \theta} = \frac{2 - \cos \theta}{2 + \sin \theta} \]
\[ 2 \sin \theta + \sin^2 \theta = 2 \cos \theta - \cos^2 \theta \]
\[ 1 = 2(\cos \theta - \sin \theta) \]
\[ \frac{1}{2} = \sqrt{2} \cos (\theta + 45^\circ) \]
\[ \cos (\theta + 45^\circ) = \frac{1}{\sqrt{2}} \]
\[ \therefore \theta + 45^\circ = 69.2952^\circ \text{ or } 290.7048^\circ \text{ (rejected) } \]
\[ \therefore \theta = 24^\circ \quad (\text{cor. to nearest degree}) \]

14) \( C_1 \) and \( C_2 \) are circles \( x^2 + y^2 = 36 \) and \( x^2 + y^2 - 10x + 16 = 0 \) respectively.

a) i) Show that, for all values of \( \theta \), the variable point \( P \) \( (6 \cos \theta, 6 \sin \theta) \) always lies on \( C_1 \).
ii) Find, in terms of \( \theta \), the equation of the tangent to \( C_1 \) at \( P \) \( (6 \cos \theta, 6 \sin \theta) \).

(3 marks)
b) Let $L$ be the common tangent to $C_1$ and $C_2$ with positive slope (See Figure 7)

i) Using (a), or otherwise, find the equation of $L$.

ii) It is known that $C_1$ and $C_2$ intersect at two distinct points $Q$ and $R$. A circle $C_3$, passing through $Q$ and $R$, is bisected by $L$. Find the equation of $C_3$.

(9 marks)

**Answer:**

ai) 
Sub $P$ into $C_1$:

$LHS = (6\cos \theta)^2 + (6\sin \theta)^2$

$= 36$

$= RHS$

$\therefore P$ is on $C_1$.

ii) 
The tangent:

$x(6\cos \theta) + y(6\sin \theta) - 36 = 0$

$x \cos \theta + y \sin \theta - 6 = 0$
Center of $C_2 = (5, 0)$
Radius of $C_2 = \frac{1}{2} \sqrt{(-10)^2 - 4(16)} = 3$

\[ \therefore L \text{ is tangent to } C_1 \]
\[ \therefore \text{Let the equation of } L \text{ be } x \cos \theta + y \sin \theta - 6 = 0 \]
\[ \therefore L \text{ is tangent to } C_2 \]
\[ \therefore \text{Radius of } C_2 = \text{Distance from center of } C_2 \text{ to } L \]
\[ \therefore 3 = \frac{5 \cos \theta - 6}{\sqrt{\cos^2 \theta + \sin^2 \theta}} \]
\[ \therefore \pm 3 = 5 \cos \theta - 6 \]
\[ \therefore \cos \theta = \frac{3}{5} \text{ or } \frac{8}{5} \text{ (rejected)} \]
\[ \therefore \text{Slope of } L > 0 \]
\[ \therefore -\frac{\cos \theta}{\sin \theta} > 0 \]
\[ \therefore \sin \theta < 0 \]
\[ \therefore \sin \theta = -\sqrt{1 - \left(\frac{3}{5}\right)^2} = -\frac{4}{5} \]
\[ \therefore L: \]
\[ \frac{3}{5} x - \frac{4}{5} y - 6 = 0 \]
\[ 3x - 4y - 30 = 0 // \]
ii) The family of circles passing through $Q$ and $R$:
\[ k(x^2 + y^2 - 36) + (x^2 + y^2 - 10x + 16) = 0 \]
\[ (1 + k)x^2 + (1 + k)y^2 - 10x + (16 - 36k) = 0 \]
Where $k$ is an arbitrary constant
\[ \therefore C_3 \text{ is in this family} \]

\[ \therefore L \text{ bisects } C_3 \]
\[ \therefore \text{Center of } C_3 \text{ is on } L \]
\[ \therefore \left( \frac{5}{1+k}, 0 \right) \text{ is on } L \]
\[ \therefore 3 \left( \frac{5}{1+k} \right) - 30 = 0 \]
\[ \frac{5}{1+k} = 10 \]
\[ k = -\frac{1}{2} \]
\[ \therefore C_3 : \]
\[ (1 - \frac{1}{2})x^2 + (1 - \frac{1}{2})y^2 - 10x + (16 + 36(\frac{1}{2})) = 0 \]
\[ \frac{1}{2}x^2 + \frac{1}{2}y^2 - 10x + 34 = 0 \]
\[ x^2 + y^2 - 20x + 68 = 0 \]

15) Given two curves $C_1: y = f(x)$, where $f(x)$ is a quadratic function, and
$C_2 : y = -\frac{1}{5} x^2 - \left( \frac{h - 20}{10} \right) x + h.$
$C_1$ has the vertex $(4, 9)$ and passes through the point $(10, 0)$.

a) Show that $f(x) = -\frac{1}{5} x^2 + 2x + 5$. 

b) i) Show that $C_2$ also passes through the point $(10, 0)$
ii) If $C_1$ and $C_2$ meet at two points, find, in terms of $h$, the x-coordinate of the point other than $(10, 0)$.
c) Figure 8 shows a fountain. A vertical water pipe $OP$ of height 15 units is installed on the horizontal ground. Two streams of water are ejected continuously from two small holes $D_1$ and $D_2$ in the pipe, with $D_2$ above $D_1$. The two streams of water lay in the same vertical plane. A rectangular coordinate system is introduced in this plane, with $O$ as the origin and $OP$ on the positive $y$-axis. The formula is designed such that the stream of water ejected from $D_1$ lies on the curve $C_1$, and that ejected from $D_2$ lies on $C_2$.

i) Find $OD_1$.

ii) If the two streams of water do not cross each other in the air before meeting at the same point on the ground, find the range of possible values of $OD_2$.

(4 marks)
Answer:

a)  
\[ \therefore C_1 \text{ has the vertex } (4,9) \]
\[ \therefore C_1 \text{ is symmetric about } x = 4 \]
\[ \therefore C_1 \text{ also passes through } (2(4) - 10, 0) = (-2, 0) \]

Let \( C_3 \) be the curve \( y = g(x) \) such that \( g(x) = -\frac{1}{4}x^2 + 2x + 5 \)
\[ \therefore g(4) = -\frac{1}{4}(4)^2 + 2(4) + 5 = 9 \]
and \( g(10) = -\frac{1}{4}(10)^2 + 2(10) + 5 = 0 \)
and \( g(-2) = -\frac{1}{4}(-2)^2 + 2(-2) + 5 = 0 \)
\[ \therefore C_3 \text{ passes through } (4,9), (10,0) \text{ and } (-2,0) \]
But since a quadratic function curve that passes through three given general points is unique
\[ \therefore C_1 \text{ and } C_3 \text{ is the same curve} \]
\[ \therefore f(x) = g(x) = -\frac{1}{4}x^2 + 2x + 5 \]

b)  
When \( x = 10, C_2 : \)
\[ y = -\frac{1}{5}(10)^2 - \left( \frac{h - 20}{10} \right)(10) + h \]
\[ y = -20 - h + 20 + h \]
\[ y = 0 \]
\[ \therefore C_2 \text{ also passes through } (10,0) \]
ii)

∴ $C_1$ and $C_2$ meet at two points

$$\therefore -\frac{1}{4}x^2 + 2x + 5 = -\frac{1}{5}x^2 - \left(\frac{h - 20}{10}\right)x + h$$

$$\frac{1}{20}x^2 - \frac{h}{10}x + (h - 5) = 0$$

$$\therefore x = \frac{\frac{h}{10} \pm \sqrt{\left(\frac{h}{10}\right)^2 - 4\left(\frac{1}{20}\right)(h - 5)}}{2\left(\frac{1}{20}\right)}$$

$$= \frac{\frac{h}{10} \pm \sqrt{\frac{h^2}{100} - \frac{1}{2}} + 1}{\frac{1}{10}}$$

$$= h \pm \sqrt{(h - 10)^2}$$

$$\therefore x = h + h - 10 \quad \text{or} \quad x = h - h + 10 \quad \text{(rejected} \therefore x \neq 10)$$

$$= 2h - 10$$

∴ The $x$-coordinate of the point is $2h - 10$.

ci)

$OD_1 = y$-intercept of $C_1$

$$= 5$$

ii)

$OD_2 = y$-intercept of $C_2 = h$

∴ $D_2$ is above $D_1$ and $D_2$ is on $OP$

∴ $OD_1 < OD_2 < OP$

∴ $5 < h < 15 \quad \text{...(*)}$

∴ $C_1$ do not cross each other in the air

∴ The the other intersection of $C_1$ and $C_2$ should be on the right of $(10,0)$

∴ $2h - 10 > 10$

$$2h > 20$$

$$h > 10$$

∴ $(*)$:

$$10 < h < 15$$

∴ Range of $OD_2$ is $10 < OD_2 < 15$.
In Figure 9, \(ABCD\) is a quadrilateral inscribed in a circle centered at \(O\) and with radius \(r\), such that \(AB \parallel DC\) and \(O\) lies inside the quadrilateral. Let \(\angle COD = 2\theta\) and reflex \(\angle AOB = 2\beta\), where \(0 < \theta < \frac{\pi}{2} < \beta < \pi\). Point \(E\) denotes the foot of perpendicular from \(O\) to \(DC\). Let \(S\) be the area of \(ABCD\).

a) Show that 
\[ S = \frac{r^2}{4} \left( \sin 2\theta - \sin 2\beta + 2 \sin (\beta - \theta) \right). \]

(3 marks)

b) Suppose \(\beta\) is fixed. Let \(S_\beta\) be the greatest value of \(S\) as \(\theta\) varies.

Show that 
\[ S_\beta = 2r^2 \sin^3 \frac{2\beta}{3} \] and the corresponding value of \(\theta\) is \(\frac{\beta}{3}\).

[Hint: You may use the identity \(\sin 3a = 3 \sin a - 4 \sin^3 a\).]

(6 marks)

c) A student says:
Among all possible values of \(\beta\), the quadrilateral \(ABCD\) becomes a square when \(S_\beta\) in (b) attains its greatest value.
Determine whether the student is correct or not.

(3 marks)
Answer:

a) 
\[ \angle AOB = 2\pi - 2\beta \]

\[ \therefore \text{Area of } \triangle AOB = \frac{1}{2} r^2 \sin (2\pi - 2\beta) = -\frac{1}{2} r^2 \sin 2\beta \]

Area of \( \triangle AOD = \frac{1}{2} r^2 \sin (\beta - \theta) \)

Area of \( \triangle BOC = \frac{1}{2} r^2 \sin (\beta - \theta) \)

Area of \( \triangle DOC = \frac{1}{2} r^2 \sin 2\theta \)

\[ \therefore S = -\frac{1}{2} r^2 \sin 2\beta + \frac{1}{2} r^2 \sin (\beta - \theta) + \frac{1}{2} r^2 \sin (\beta - \theta) + \frac{1}{2} r^2 \sin 2\theta \]

\[ = \frac{1}{2} (\sin 2\theta - \sin 2\beta + 2\sin (\beta - \theta)) \]

b) 
Let \( S(\theta) \) be the value of \( S \) with fixed \( \beta \) and the corresponding \( \theta \)

\[ \therefore S'(\theta) = \frac{d}{d\theta} \left( \frac{r^2}{2} (\sin 2\theta - \sin 2\beta + 2\sin (\beta - \theta)) \right) \]

\[ = \frac{d}{d\theta} \left( \frac{r^2 \sin 2\theta}{2} - \frac{r^2 \sin 2\beta}{2} + r^2 \sin (\beta - \theta) \right) \]

\[ = \frac{2r^3 \cos 2\theta}{2} - r^2 \cos (\beta - \theta) \]

\[ = r^2 (\cos 2\theta - \cos (\beta - \theta)) \]

\[ S''(\theta) = \frac{d}{d\theta} \left( r^2 (\cos 2\theta - \cos (\beta - \theta)) \right) \]

\[ = -2r^2 \sin 2\theta - r^2 \sin (\beta - \theta) \]

\[ = -r^2 (2 \sin 2\theta + \sin (\beta - \theta)) \]
\[ S(\theta) \text{ attains is maximum when } S'(\theta) = 0 \text{ and } S''(\theta) < 0 \]

\[ 0 = r^2 \left( \cos 2\theta - \cos (\beta - \theta) \right) \]

\[ \cos 2\theta = \cos (\beta - \theta) \]

\[ 2\theta = \beta - \theta \quad (\theta < \pi) \]

\[ \theta = \frac{\beta}{3} \]

\[ S'' \left( \frac{\beta}{3} \right) = -r^2 \left( 2 \sin \frac{2\beta}{3} + \sin (\beta - \frac{\pi}{2}) \right) \]

\[ = -3r^2 \sin \frac{2\beta}{3} \]

\[ \therefore \frac{\pi}{2} < \beta < \pi \]

\[ \frac{\pi}{3} < \frac{2\beta}{3} < \frac{2\pi}{3}, \text{ i.e., } \frac{2\beta}{3} \text{ is in quadrant I} \]

\[ \therefore \sin \frac{2\beta}{3} > 0 \]

\[ \therefore S'' \left( \frac{\beta}{3} \right) < 0 \]

\[ \therefore S \text{ is maximum when } \theta = \frac{\beta}{3} \]

\[ S_{\beta} = \frac{\nu^{2}}{2} \left( \sin 2\theta - \sin 2\beta + 2 \sin (\beta - \theta) \right) \]

\[ = \frac{\nu^{2}}{2} \left( \sin 2\theta - \sin 6\theta + 2 \sin 2\theta \right) \]

\[ = \frac{\nu^{2}}{2} \left( 3 \sin 2\theta - 3 \sin 2\theta + 4 \sin^3 2\theta \right) \]

\[ = 2r^2 \sin^3 2\theta \]

\[ = 2r^2 \sin^3 \frac{2\beta}{3} \]

c)

Yes.

\[ S_{\beta} \text{ is maximum when } \sin^3 \frac{2\beta}{3} = 1 \]

\[ \therefore \text{The corresponding value of } \frac{2\beta}{3} = \frac{\pi}{2} \]

\[ \therefore \beta = \frac{\pi}{4} \]

\[ \therefore \theta = \frac{\beta}{3} = \frac{\pi}{6} \]

\[ \therefore \angle DOC = \frac{\pi}{2} \]

\[ \therefore BOD \text{ and } AOC \text{ are perpendicular straight lines} \]

But \[ BOC = AOC = 2r \]

\[ \therefore ABCD \text{ is a square} \]

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17a) Let \( y = (x - \pi) \sin x + \cos x \).

i) \[ \text{Show that } \frac{dy}{dx} = (x - \pi) \cos x \]

Hence find \[ \int (x - \pi) \cos x \, dx. \]
ii) Figure 10 shows the graphs of $y = (x - \pi) \sin x + \cos x$ for $0 \leq x \leq \frac{3\pi}{2}$.

\[ y = (x - \pi) \cos x \]

![Figure 10](image)

1) Find the areas of the two shaded regions $R_1$ and $R_2$ as shown in Figure 10.

2) Find $\int_{\pi/2}^{3\pi/2} (x - \pi) \cos x \, dx$ \hspace{1cm} (7 marks)

b)

Let $f(x)$ be a continuous function. Figure 11 shows a sketch of the graph of $y = f'(x)$ for $0 \leq x \leq x_4$. It is known that the areas of the shaded regions $S_1$ and $S_2$ as shown in Figure 11 are equal.

i) Show that $f(x_1) = f(x_3)$

ii) Furthermore, $f(0) = f(x_4) = 0$ and $f(x) \neq 0$ for $0 < x < x_4$. In Figure 12, draw a sketch of the graph of $y = f(x)$ for $0 \leq x \leq x_4$. \hspace{1cm} (5 marks)
\textbf{Answer:}
\begin{enumerate}
\item \( \frac{dy}{dx} = \frac{d}{dx} \left( (x - \pi) \sin x + \cos x \right) \)
\[ = \frac{d}{dx} \left( (x - \pi) \sin x \right) - \sin x \]
\[ = (x - \pi) \cos x + \sin x - \sin x \]
\[ = (x - \pi) \cos x, \]
\[ \therefore \int (x - \pi) \cos x \, dx = \int \frac{dy}{dx} \, dx \]
\[ = y + C \]
\[ = (x - \pi) \sin x + \cos x + C \quad \text{for some constant } C_i \]
\item \( R_1 = \int_{\pi/2}^{\pi} (x - \pi) \cos x \, dx \)
\[ = (x - \pi) \sin x + \cos x \bigg|_{\pi/2}^{\pi} \]
\[ = \cos \pi + \frac{\pi}{2} \sin \frac{\pi}{2} - \cos \frac{\pi}{2} \]
\[ = -1 + \frac{\pi}{2} - 0 \]
\[ = \frac{\pi}{2} - 1_i \]
\[ R_2 = \left| \int_{\pi}^{3\pi/2} (x - \pi) \cos x \, dx \right| \]
\[ = \left| (x - \pi) \sin x + \cos x \right|_{\pi}^{3\pi/2} \]
\[ = \frac{3\pi}{2} \sin \frac{3\pi}{2} + \cos 3\pi - \cos \pi \]
\[ = -\frac{3\pi}{2} + 0 - 1 \]
\[ = -\frac{3\pi}{2} - 1 \]
\[ = \frac{\pi}{2} - 1_i \]
\item \[ \int_{\pi/2}^{3\pi/2} (x - \pi) \cos x \, dx = \int_{\pi/2}^{\pi} (x - \pi) \cos x \, dx + \int_{\pi}^{3\pi/2} (x - \pi) \cos x \, dx \]
\[ = \frac{\pi}{2} - 1 + 1 - \frac{\pi}{2} \]
\[ = 0_i \]
\end{enumerate}
i) $S_1 = S_2$

\[ \int_{x_1}^{x_2} f'(x) \, dx = -\int_{x_2}^{x_3} f'(x) \, dx \]
\[ f(x_2) - f(x_1) = -f(x_3) + f(x_2) \]
\[ f(x_1) = f(x_3) \]

ii) $f(x) = f(x_\parallel)$

Figure 12