第十八屆聯校數學競賽選拔賽
The 18th Inter-School Mathematics Preliminary Contest

指引
Instructions:
1. 本比賽限時兩小時.
   The time limit for this contest is 2 hours.
2. 比賽滿分為 100 分.
   The total score is 100.
3. 比賽其間不得使用計算機.
   Calculator is not allowed in the contest.
4. 除特別指明外，問題中所有數字皆為實數，並以十進制表達.
   Unless otherwise specified, all numbers appeared in the questions are real numbers, and are in decimal system.
5. 除特別指明外，所有答案必須以其正確值表達，並約至最簡．
   Unless otherwise specified, all answers must be given in the simplest form.
   Approximation is not allowed.
6. 若題目完全答對則可得該題分數，答錯或棄權不扣分．
   Marks will be given if the question is answered totally correct. Marks will not be deducted if the answer is wrong or blanked.
7. 比賽其間不得作弊，否則取消其參賽資格．
   Cheating is prohibited, or the contest will be disqualified.
8. 得分最高四名可代表本校參加第十八屆聯校數學競賽．
   The four contesters with highest scores will be the representatives of this school for the 18th inter-school mathematics contest.
1) 定義 \( x_0 = 1 \), \( x_{n+1} = 2 - x_n^2 \). 求 \( \prod_{k=0}^{200400} x_k \).  
(2 分)

Define \( x_0 = 1 \), \( x_{n+1} = 2 - x_n^2 \). Find \( \prod_{k=0}^{200400} x_k \).  
(2 marks)

2) \( 2004^{2004} \) 除以 7 的餘數是？
What is the remainder of \( 2004^{2004} \) when it is divided by 7？  
(3 分)

3) 在一坐標平面上，A 和 B 的坐標分別為 (1, 2) 和 (3, 1). 若 L 為 AB 的垂直平分線，C 在 L 上，且 AC + BC = AB. 求 C 的坐標.
On a coordinates plane, the coordinates of A and B are (1, 2) and (3, 1) respectively. If L is the perpendicular bisector of AB, C is on L, and AC + BC = AB, find the coordinates of C.  
(3 分)

4) 求所有的 \( n \in \mathbb{N} \) 使得 \( n | 2n + 20 \).
Find all \( n \in \mathbb{N} \) such that \( n | 2n + 20 \).  
(3 分)

5) 若 \( A, B, C, D \in \mathbb{Z} \)，且 \( 5^4 + 5^8 + 5^C + 5^D = 150.208 \)，求 \( A, B, C, D \) 的值.
If \( A, B, C, D \in \mathbb{Z} \)，and \( 5^4 + 5^8 + 5^C + 5^D = 150.208 \)，find the values of \( A, B, C, D \).  
(4 分)

6) 若一個數 \( n \) 能被 6 整除，而除 9 的餘為 6，則我們稱之為「好數」。例如，2004 便是一個「好數」。求 2000 至 3000 之間的「好數」個數。
If a number \( n \) is divisible by 6, and the remainder is 6 when divided by 9, we call this a “good number”. For example, 2004 is one of them. How many “good numbers” are between 2000 and 3000?  
(4 分)

7) 定義 \( i = \sqrt{-1} \). 求 \( x^3 + (1 - i)x^2 + (2 - i)x + 2 = 0 \) 的所有解。
(提示：其中一個解為整數).  
Define \( i = \sqrt{-1} \). Solve \( x^3 + (1 - i)x^2 + (2 - i)x + 2 = 0 \).  
(Hint: One of the solutions is an integer.)  
(4 分)

8) 求 \( x^2 + y^2 = 2(x + y) \) 的所有整數解。
Find all integral solutions to \( x^2 + y^2 = 2(x + y) \).  
(5 分)
9) 求 $24!$ 的正因數個數.  
Find the number of positive factors of $24!$.  

10) 在 $\triangle ABC$ 中，$H$ 是其垂心. 延長 $BH$ 交 $AC$ 於 $E$. 畫一以 $H$ 為圓心、$EH$ 為半徑的圓. 假設 $B$ 剛好在圓形上，又 $AB$ 交圓於 $F$. 已知 $EF = AF = \frac{1}{2} BC = 2004$，求 $BF$ 的長度.  
In $\triangle ABC$, $H$ is the orthocenter. Extend $BH$ to intersect $AC$ at $E$. Construct a circle with center $H$ and radius $EH$. Assume $B$ is on the circle, and $AB$ intersects the circle at $F$. Given that $EF = AF = \frac{1}{2} BC = 2004$, find the length of $BF$.  

11) 若 $a$, $b$ 和 $c$ 都是任意小於 10 的正整數，求 $ab + c$ 為偶數的概礎.  
If $a$, $b$ and $c$ are some integers which are less than 10, find the probability such that $ab + c$ is even.  

12) 若甲先生約會丁小姐的概率為 $\frac{1}{7}$、乙先生約會丁小姐的概率為 $\frac{2}{7}$、丙先生約會丁小姐的概率為 $\frac{4}{7}$，求丁小姐有約會的概率.  
If the probability of Ms. $D$ dated by Mr. $A$ is $\frac{1}{7}$, Mr. $B$ is $\frac{2}{7}$ and Mr. $C$ is $\frac{4}{7}$, find the probability that Ms. $D$ is having a date.  

13) 求 $\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots$ 的值.  
Find $\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots$  


15) 設 $\triangle ABC$ 是平面上的一個三角形.$D$ 為 $A$ 點沿 $BC$ 的反射、$E$ 為 $B$ 點沿 $AC$ 的反射、$F$ 為 $C$ 點沿 $AB$ 的反射. 若 $20AB = 15BC = 12AC$，且 $\triangle ABC$ 的面積是 $s$，求 $\triangle DEF$ 的面積.  
Let $\triangle ABC$ be a planar triangle. $D$ is the reflection of $A$ along $BC$, $E$ is the reflection of $B$ along $AC$, and $F$ is the reflection of $C$ along $AB$. If $20AB = 15BC = 12AC$, and the area of $\triangle ABC$ is $s$, find the area of $\triangle DEF$.  


16) 設 $ABCD$ 是一個四邊形，$M, N, P, Q$ 分別是 $AB, BC, CD$ 和 $AD$ 的中點，且 $X$是 $MP$ 和 $NQ$ 的交點。若 $\triangle MXP$ 的面積是 $s$，求 $ABCD$ 的面積。 (6 分)

Let $ABCD$ be a quadrilateral. $M, N, P, Q$ are midpoints of $AB, BC, CD$ and $AD$ respectively. Moreover, $X$ is the intersection of $MP$ and $NQ$. If the area of $\triangle MXP$ is $s$, find the area of $ABCD$. (6 marks)

17) 若 $\alpha, \beta$ 和 $\gamma$ 都是 $x^3 + 2x^2 + 3x + 4 = 0$ 的根，求 $\alpha^2 + \beta^2 + \gamma^2$。 (6 分)

If $\alpha, \beta$ and $\gamma$ are roots of $x^3 + 2x^2 + 3x + 4 = 0$, find $\alpha^2 + \beta^2 + \gamma^2$. (6 marks)

18) 設 $\triangle ABC$ 的外接圓圓心為 $O$. 在 $A$ 作切線 $L_1$、在 $B$ 作切線 $L_2$，使 $L_1$ 和 $L_2$ 交於 $P$. 若有 $OA = r, \angle ACB = 30^\circ$ 及 $\angle CAO = \angle PCB$，求 $BC$ 的長度。 (6 分)

Let the circumcenter of $\triangle ABC$ be $O$. Draw tangents $L_1$ and $L_2$ at $A$ and $B$ respectively. $L_1$ and $L_2$ intersect at $P$. If $OA = r, \angle ACB = 30^\circ$ and $\angle CAO = \angle PCB$, find the length of $BC$. (6 marks)

19) 小明在畫圓形。他首先畫一條長度為 2004 的直線 $AX$，然後以 $A$ 爲圓心，$AX$ 爲半徑畫了一個圓形。之後，他再以 $AX$ 的中心點 $B$ 爲圓心，$BX$ 爲半徑畫一個圓。接著他又取 $BX$ 的中點 $C$ 爲圓心，$CX$ 爲半徑畫一圓。假設他不斷重覆這些步驟，並畫了無限多個圓形，問他畫了多長的線？ (7 分)

Derek is drawing circles. Firstly, he draws a straight line $AX$ with length 2004. Then, he draws a circle with center $A$ and radius $AX$. After that, he constructs a midpoint $B$ of $AX$ and draws a circle with center $B$ and radius $BX$. He then does the same thing by making a midpoint $C$ of $BX$ and draws a circle with center $C$ and radius $CX$. If he repeats the above process forever, and draws infinite circles, what is the length of the lines he has drawn? (7 marks)

20) 設 $C_1$ 和 $C_2$ 是兩個圓，其圓心分別為 $O_1$ 和 $O_2$，且 $C_1$ 過 $O_2$ 及 $C_2$ 過 $O_1$。分別以 $O_1$ 和 $O_2$ 為圓心再作兩圓 $I_1$ 和 $I_2$，使得兩圓相切及半徑相同。若 $C_1$ 交 $I_2$ 於 $A$ 和 $B$，求 $\sin \angle AO_1B$。 (9 分)

Assume $C_1$ and $C_2$ are two circles, with centers $O_1$ and $O_2$ respectively, and both of them crosses the opposite’s center. Construct two more circles $I_1$ and $I_2$ by using $O_1$ and $O_2$ as their centers, such that they tangent to each other and have the same radius. If $C_1$ and $I_2$ intersect at $A$ and $B$, find $\sin \angle AO_1B$. (9 marks)
姓名: ____________________

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總分: ________________
Answers

1) 1
2) 1
3) \( (2, \frac{3}{2}) \)
4) 1, 2, 4, 5, 10, 20
5) \(-3, -1, 2, 3\)
6) 56
7) -1, \(-i, 2i\)
8) \((0, 0), (0, 2), (2, 0), (2, 2)\)
9) 242880
10) \(501\pi\sqrt{2}\)
11) \(\frac{349}{779}\)
12) \(\frac{253}{343}\)
13) 2
14) \(\frac{\sqrt{2}}{\sqrt{3}}\)
15) 3s
16) 8s
17) \(-2\)
18) \(\sqrt{2 + \sqrt{3}}\)
19) \(8016\pi + 2004\)
20) \(\frac{\sqrt{75}}{32}\)
1) 圖中，C₁, C₂及C₃是同圓，O₁, O₂和O₃分別是它們的圓心。C₄是一個同時內切於C₁, C₂和C₃的圓。若C₁, C₂和C₃的半徑為r，求C₄的半徑。 (6分)

In the figure, C₁, C₂ and C₃ are equal circles with centers O₁, O₂ and O₃ respectively. C₄ is a circle which tangents to these three circles internally. If the radii of C₁, C₂ and C₃ are r, find the radius of C₄. (6 marks)

2) 求一個邊長為x的正四面體的體積。 (6分)

Find the volume of a regular tetrahedron which length of each side is x. (6 marks)

3) 設T為一個三角形，其中它的邊長是1, a 和b。若a和b是$x^2 + cx + 1 = 0$的根，而c為任意實數，求T的最大可能面積，及對應的c的值。 (13分)

Assume T is a triangle with side lengths 1, a and b, where a and b are roots of $x^2 + cx + 1 = 0$ and c is an arbitrary real constant. Find the largest possible area of T and the corresponding value(s) of c. (13 marks)
**Marking Scheme**

1) Acceptable answers: \( r \left( 1 - \frac{\sqrt{1}}{3} \right) \) or \( r \left( 1 - \frac{1}{\sqrt{3}} \right) \)

*Knowing the center of \( C_4 \), \( O_4 \), is the “center” of \( \Delta O_1 O_2 O_3 \) … **2 marks**
*Knowing how to calculate \( O_1 O_4 \) (or \( O_2 O_4 \), \( O_3 O_4 \)) … **2.5 marks**
*Obtaining the correct value of \( O_1 O_4 \) … **0.5 marks**
Giving the correct answer of the radius of \( C_4 \) … **1 mark**

*: If one can get \( O_1 O_4 \) using methods other than knowing \( O_4 \) is the center of \( \Delta O_1 O_2 O_3 \), also give him/her **5 marks** according to the performance.

2) Acceptable answer: \( \frac{\sqrt{2}}{6} x^3 \)

Using the formula “Volume = Base × Height ÷ 3” … **0.5 marks**
Calculating the base area (Area of a triangle) … **1.5 marks**
Calculating the height … **3 marks**
Giving the correct answer of volume … **1 mark**

3) Acceptable answer: \( \text{Area} = \frac{\sqrt{5}}{4} ; \ c = -2 \)

Using Heron’s Formula (or cosine law and \( \frac{1}{2} ab \sin \theta \), if applicable) … **1 mark**
Knowing \( a + b = -c \) and \( ab = 1 \) … **1 mark**
Finding the range of \( c \) as \( a, b \) are real and positive (\( c \leq -2 \)) … **1.5 marks**

Expressing the area of \( T \) as \( \frac{1}{2} \sqrt{-c^4 + 6c^2 - 5} \) … **4 marks**
Finding the range of \( c \) to make \( T \) a “real” triangle (\( -\sqrt{5} \leq c \leq -1 \)) … **2.5 marks**.
Combining the “max value” of \( c \) (\(^2\)) and range of \( c \) to give \( c = -2 \) … **2 marks**
Giving the correct answer of area … **1 mark**

*: Plotting the graph \( -c^4 + 6c^2 - 5 \) is also acceptable.