Mathematics High Achievers Selection 2004

Redacted by Match Chu
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Selection contest for mathematics high achievers for secondary one to four students.
# Contents

0  Regulations and Instructions 4

1  Secondary One, Questions 5
   1.1 Part A ................................................. 5
   1.2 Part B ................................................. 5
   1.3 Part C ................................................. 6

2  Secondary Two, Questions 7
   2.1 Part A ................................................. 7
   2.2 Part B ................................................. 7
   2.3 Part C ................................................. 8
   2.4 Part D ................................................. 8

3  Secondary Three, Questions 10
   3.1 Part A ................................................. 10
   3.2 Part B ................................................. 10
   3.3 Part C ................................................. 11

4  Secondary Four, Questions 12
   4.1 Part A ................................................. 12
   4.2 Part B ................................................. 13
   4.3 Part C ................................................. 13
   4.4 Part D ................................................. 14
   4.5 Part E ................................................. 14

5  Secondary One, Solutions 15
   5.1 Outline of Solutions ................................. 15
   5.2 Solutions with Full Steps ............................ 16
      5.2.1 Q1 ................................................. 16
      5.2.2 Q2 ................................................. 16
      5.2.3 Q3 ................................................. 16
      5.2.4 Q4 ................................................. 17
      5.2.5 Q5 ................................................. 17
      5.2.6 Q6 ................................................. 17
9 Statistical Information

10 Nominated Contestants

10.1 Secondary One

10.2 Secondary Two

10.3 Secondary Three

10.4 Secondary Four

11 Comments

11.1 Secondary One

11.2 Secondary Two

11.3 Secondary Three

11.4 Secondary Four
Chapter 0

Regulations and Instructions

• The competition lasts for an hour.
• The paper contains different parts with different scores and difficulty.
  – For secondary one and three contestants, the paper contains three parts: A, B and C. All questions must be attempted, but working steps are not required.
  – For secondary two contestants, the paper contains four parts: A, B, C and D. All of them are short questions and therefore working steps are not required. All problems must be attempted except part D, which is optional.
  – For secondary four contestants, the paper contains five parts: A to E. The questions in part A to D are short questions, while those in part E are long. Answers are the only requirement for short questions. However, full steps are needed in answering long questions. All problems must be attempted except part D, which is optional.
• Part D (if there is) will be scored if and only if the contestant makes 50 marks or more in the other parts.
• Use of calculators is prohibited during the competition.
• Unless otherwise specified, all answers must be exact and in their simplest form.
• Unless otherwise specified, all numbers in the questions are decimal numbers.
Chapter 1

Secondary One, Questions

1.1 Part A

Each answer is worth 7 marks.

1. It is known that when $2^n$ is divided by 10 for an integer $n$, the remainder is 6. Find the value of $n$ closest to 1000.

2. Find the value of $1 + 2 - 3 - 4 + 5 + 6 - 7 + \ldots - 999 - 1000$.

1.2 Part B

Each answer is worth 10 marks.

3. It is known that $a$ and $b$ are integers. Find the probability that $ab - a - b$ is even.

4. Find the remainder of $111^{1111}$ when it is divided by 11.

5. If the sum of all interior angles of an $n$-sided polygon is a perfect square in degrees, find the minimum of $n$.

6. It is known that the area of a square is same as a circle. If the perimeter of that square is 1 cm, find the perimeter of the circle.

7. In figure 1.1, it is known that $CD = 3$, $BC = 4$, $CE = 4.5$ and $AC = 6$. Find the height of the trapezoid $ABDE$. 
1.3 Part C

Each answer is worth 12 marks.

8. Find the value of the following expression:

$$\left( \left( 3 + \frac{1}{4} \right)^2 - 3^2 - \frac{1}{4^2} \right) \times \left( \left( 8 + \frac{1}{9} \right)^2 - 8^2 - \frac{1}{9^2} \right) \times \left( \left( 15 + \frac{1}{16} \right)^2 - 15^2 - \frac{1}{16^2} \right) \times \cdots \times \left( \left( 99 + \frac{1}{100} \right)^2 - 99^2 - \frac{1}{100^2} \right)$$

9. Find the integral solutions to $11x + 13y = 5$ such that the values of $x$ and $y$ are the nearest.

10. Suppose there is a positive integer $n$ such that the remainders of it are 1, 1, 2 and 3 when it is divided by 2, 3, 5 and 7 respectively. Find the minimum value of $n$. 
Chapter 2

Secondary Two, Questions

2.1 Part A

Each answer is worth 6 marks.

1. Three different integers are chosen between 1 and 9 inclusively. Find the probability that the sum of these three numbers is an odd number.

2. Find all integral solutions to $6^2 + x^2 = y^2$.

2.2 Part B

Each answer is worth 9 marks.

3. Solve the following simultaneous equations:
   \[
   \begin{cases}
   xy + x + y = 1 \\
   yz + y + z = 5 \\
   zx + z + x = 2
   \end{cases}
   \]

4. If $a$ and $b$ are positive integers and $a < b < 100$, find the number of triangles that the lengths of the three sides are $a, b$ and 100.

5. It is known that $ABCD$ is a trapezoid with $AB \parallel CD$ and $AB \perp BC$. The straight lines $AC$ and $BD$ intersect at $X$. If $AB = 9$, $BC = 12$ and $CD = 16$, find the area of $\triangle BXC$.
6. Find the smallest positive integer \( x \) such that \( 4^{999} + 4^{1111} + 4^x \) is a perfect square.

2.3 Part C

Each answer is worth 13 marks.

7. Let \( k \) be a real number. If \( \alpha \) and \( \beta \) are two real roots of \( x \) in the equation \( x^2 + (k - 2)x + (k^2 + 3k + 5) = 0 \), find the maximum of \( \alpha^2 + \beta^2 \).

8. Find \( \frac{x^3}{x^2 + x + 1} \) if \( x + \frac{1}{x} = 4 \).

9. Find the sum of all fractions of the form \( \frac{1}{(n+1)m+n} \), where \( m \) and \( n \) are positive integers.

10. Find the minimum of \( 3x^2 + 2y^2 + 2z^2 + 4xy + 2yz + 2zx - 4x - 4y - 8z \) if \( x + y + z = 2 \).

2.4 Part D

Each answer is worth 20 marks; contestants may ignore this part.

In polar coordinates system there is a curve \( S : r = 1 + \frac{\theta}{2\pi} \). \( A(1,0) \) and \( B(2,0) \) are two points on the curve. (See figure 2.1)

11. Find the length of the curve \( S \) from \( A \) to \( B \).

12. Find the area enclosed by the curve \( S \) from \( A \) to \( B \).
Figure 2.1: Figure to MHAS 2004/2 Q11 and Q12
Chapter 3

Secondary Three, Questions

3.1 Part A

Each answer is worth 6 marks.

1. Solve the following:
   \[ \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \cdots + \frac{1}{(x+11)(x+12)} + \frac{1}{x+12} = \frac{1}{4} \]

2. If \( a^3 = 150b \) and \( a, b \) are both positive integers, find the minimum of \( b \).

3. Three different integers are chosen between 1 and 10 inclusively. Find the probability that the product of these three numbers is a perfect square.

3.2 Part B

Each answer is worth 10 marks.

4. Find the following:
   \[ \frac{1}{2} + \frac{1}{3} + \frac{2}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \cdots + \frac{1}{100} + \frac{2}{100} + \cdots + \frac{99}{100} \]

5. In a piece of graph paper with 3 x 3 squares, each square of it is colored either by red, green or blue. It is known that the color of any pair of adjacent squares must be different. How many distinct ways can one color this graph paper?
6. Find the remainder of \(2^{3^{45 \cdots 99}}\) when it is divided by 100.

7. How many integers are there between 1 and 1000 that their decimal representation contains exactly one “2”?

8. It is known that in \(\triangle ABC\), \(AB = 2\) and \(CD = 3\). \(M\) and \(N\) are two points on \(BC\) such that \(BM = CM\) and \(\angle BAN = \angle CAN\). Find \(BN : NC\).

### 3.3 Part C

Each answer is worth 16 marks.

10. We define a sequence \(F\) as:

\[
\begin{align*}
F_0 &= F_1 = 1 \\
F_{n+1} &= F_n + F_{n-1}
\end{align*}
\]

That means \(F = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}\). Hence find the following:

\[
\sum_{k=0}^{\infty} \frac{1}{1} + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \frac{8}{32} + \frac{13}{64} + \frac{21}{128} + \frac{34}{256} + \frac{55}{512} + \ldots
\]

11. Express \(\sin 42^\circ\) as surd form.
Chapter 4

Secondary Four, Questions

4.1 Part A

Each answer is worth 8 marks.

1. Find the sum of the all marked angles referring to figure 4.1.

2. Find $\log_{1440} 2 + \log_{1440} \sqrt{5} + \log_{1440} \sqrt{5}$.

3. In figure 4.2, it is known that $AB = BC = CA = 2$ and $DE = EF = FA$. Find the area of the circle.

Figure 4.1: Figure to MHAS 2004/4 Q1
4.3 Part C

Each answer is worth 13 marks.

7. If $\cos 15^\circ = \frac{\sqrt{a} + \sqrt{b}}{4}$ for positive integers $a$ and $b$, find $a + b$.

8. Find $\frac{1}{5} + \frac{4}{5^2} + \frac{9}{5^3} + \frac{16}{5^4} + \ldots$.
4.4 Part D

Each answer is worth 15 marks; contestants may ignore this part.

9. Find
\[
\int \int_A \sqrt{1 - x^2 - y^2} \, dx \, dy
\]
Where
\[
A = \left\{ (x, y) \mid x \in [-1, 1], y \in \left[ -\sqrt{1-x^2}, \sqrt{1-x^2} \right] \right\}
\]

10. Find a complex number \( z \) such that \( \sin z = 2 \).

4.5 Part E

Each answer is worth 20 marks; please provide full steps.

11. It is known that \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty \). Hence find \( 0 + 1 - 2 \times 3 \div 4 + 5 - 6 \times 7 \div 8 + 9 - 10 \times \ldots \).
5.1 Outline of Solutions

1. 1000
2. -1000
3. \( \frac{1}{4} \)
4. 1
5. 7
6. \( \frac{\sqrt{\pi}}{2} \)
7. \( \frac{6}{5} \)
8. \( \frac{2816}{5} \)
9. \( x = 4, y = -3 \)
10. 157
5.2 Solutions with Full Steps

5.2.1 Q1

The answer is 1000.

Consider \( n = 1, 2, 3, 4, 5, \ldots \). We will find that the remainder of \( 2^n \) divided by 10 is 2, 4, 8, 6, 2, \ldots. Observe that the pattern repeats for every four numbers. Therefore, the remainder of \( 2^{1000} \) is the same as \( 2^4 \), which is 6. Thus \( n = 1000 \).

5.2.2 Q2

The answer is \(-1000\).

\[
1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \cdots - 999 - 1000 = \underbrace{(1 + 2 - 3 - 4)} + \underbrace{(5 + 6 - 7 - 8)} + \cdots + \underbrace{(997 + 998 - 999 - 1000)}_{250} \\
= (-4) + (-4) + \cdots + (-4)_{250} \\
= 250 \times -4 \\
= -1000
\]

5.2.3 Q3

The answer is \( \frac{1}{4} \). 0.25 is also acceptable.

Consider the parity of \( a \) and \( b \). There will be a total of 4 possible outcomes. We will get the following:

<table>
<thead>
<tr>
<th></th>
<th>b</th>
<th>( ab - a - b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>Even</td>
<td>Even</td>
</tr>
<tr>
<td>Odd</td>
<td>Even</td>
<td>Odd</td>
</tr>
<tr>
<td>Even</td>
<td>Odd</td>
<td>Odd</td>
</tr>
<tr>
<td>Odd</td>
<td>Odd</td>
<td>Odd</td>
</tr>
</tbody>
</table>

Out of the 4 outcomes, there are only 1 that result in an even number. Therefore, the probability is \( \frac{1}{4} \).
5.2.4 Q4

The answer is 1.

Since the GCD of 111 and 11 is 1, we can apply Fermat’s Little Theorem:

\[ 111^{1111} \equiv (111^{10})^{111} \cdot 111 \equiv 111 \equiv 1 \pmod{11} \]

5.2.5 Q5

The answer is 7.

The sum of interior angles of an \( n \)-gon is \( 180^\circ (n - 2) \).

**Solution 1:**
By trial and error, we get when \( n = 7 \), the sum is \( 900^\circ = (30^2)^\circ \).

**Solution 2:**
Since \( 180 = 2^2 \cdot 3^2 \cdot 5 \), if this is multiplied by 5 the result will be a perfect square. Thus \( n - 2 = 5 \), implying \( n = 7 \).

5.2.6 Q6

The answer is \( \frac{\sqrt{\pi}}{2} \). \( \frac{\pi}{2\sqrt{\pi}} \) is also acceptable.

Let \( r \) be the radius of the circle and \( p \) be the side length of the square. Then \( p = \frac{1}{4} \). The area of the square is then \( p^2 = \frac{1}{16} \). Since the areas of the circle and the square are the same, we have \( \pi r^2 = \frac{1}{16} \), meaning \( r = \frac{1}{4\sqrt{\pi}} \). So the perimeter of the circle is \( 2\pi r = \frac{2\pi}{4\sqrt{\pi}} = \frac{\sqrt{\pi}}{2} \).

5.2.7 Q7

The answer is \( \frac{6}{5} \). 1.2 and 1\( \frac{1}{5} \) is also acceptable.

Let \( H \) be a point on \( BD \) such that \( CH \perp BD \). Let \( K \) be a point on \( AE \) such that \( CK \perp AE \). Then \( HK \) will be the height of the trapezoid \( ABDE \). Applying Pythagoras’ Theorem on \( \triangle BCD \) and \( \triangle ACE \), we have \( BD = \sqrt{3^2 + 4^2} = 5 \) and \( AE = \sqrt{4.5^2 + 6^2} = 7.5 \). Now, in \( \triangle BCD \),

\[ BC \cdot CD = 2(\text{Area of } \triangle BCD) = CH \cdot BD \]
Which means $CH = \frac{3 \times 4}{5} = \frac{12}{5}$. Similarly, $CK = \frac{4 \times 5 \times 6}{7 \times 5} = \frac{18}{7}$. Hence $HK = CK - CH = \frac{6}{5}$.

5.2.8 Q8

The answer is $\frac{2816}{5}$. 563.2 and $563\frac{1}{5}$ is also acceptable.

Since

$$\left( k^2 - 1 + \frac{1}{k^2} \right)^2 - (k^2 - 1)^2 - \frac{1}{k^4}$$

$$= \left( \frac{1}{k^2} \right) \left( \frac{1}{k^2} + 2 (k^2 - 1) \right) - \frac{1}{k^4}$$

$$= \left( \frac{1}{k^2} \right) (2k^2 - 2)$$

$$= 2 \left( 1 - \frac{1}{k^2} \right)$$

Therefore, the product is equal to

$$2^{10} \left( 1 - \frac{1}{4} \right) \left( 1 - \frac{1}{9} \right) \left( 1 - \frac{1}{16} \right) \cdots \left( 1 - \frac{1}{100} \right)$$

$$= 1024 \cdot 3 \cdot 8 \cdot 15 \cdot 24 \cdot 35 \cdot 48 \cdot 63 \cdot 80 \cdot 99$$

$$= \frac{2816}{5}$$

5.2.9 Q9

The answer is $x = 4, y = -3$.

Using Extended Euclidean Algorithm, we have:

<table>
<thead>
<tr>
<th>Quotient</th>
<th>Remainder</th>
<th>$y$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-5</td>
<td>6</td>
</tr>
</tbody>
</table>

Therefore, $11(6) + 13(-5) = 1$. Multiply both sides by 5 to get $11(30) + 13(-25) = 5$. But this is not the closest for $x$ and $y$. We can, however, subtract $x$ by 13 and add $y$ by 11 simultaneously and repeatly, as the overall value is..
retained but the two values are getting closer. The result is then \( x = 4, y = -3 \) by trial and error.

### 5.2.10 Q10

The answer is 157. Any value of the form \( 157 + 210k \) is also acceptable.

\[
\begin{align*}
\{ & n \equiv 1 \pmod{2} \\
& n \equiv 1 \pmod{3} \\
& n \equiv 2 \pmod{5} \\
& n \equiv 3 \pmod{7} \\
\} \\
\therefore & 105n \equiv 105 \pmod{210} \\
\therefore & 70n \equiv 70 \pmod{210} \\
\therefore & 42n \equiv 84 \pmod{210} \\
\therefore & 30n \equiv 90 \pmod{210} \\
\therefore & 247n \equiv 349 \pmod{210}
\end{align*}
\]

That means there exists an integer \( k \) such that \( 247n + 210k = 349 \). Using Extended Euclidean Algorithm, we have:

<table>
<thead>
<tr>
<th>Quotient</th>
<th>Remainder</th>
<th>( n )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>247</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>210</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>37</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>25</td>
<td>-5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>6</td>
<td>-7</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>-17</td>
<td>20</td>
</tr>
</tbody>
</table>

Therefore \( n = -17 \times 349 + 210m \) for any integers \( m \). Particularly, \( n = 157 \) is the smallest positive value.
Chapter 6

Secondary Two, Solutions

6.1 Outline of Solutions

1. $\frac{10}{\pi}$
2. $(0, 6), (0, -6), (8, 10), (-8, 10), (8, -10), (-8, -10)$
3. $(0, 1, 2), (-2, -3, -4)$
4. 2401
5. $\frac{864}{25}$
6. 886
7. 18
8. $\frac{1}{\sqrt{3}}$
9. 1
10. $-8$
11. $\sqrt{1 + 16\pi^2} - \frac{1}{2} \sqrt{1 + 4\pi^2} + \frac{\sinh^{-1} 4\pi - \sinh^{-1} 2\pi}{4\pi}$
12. $\frac{14\pi}{3}$
6.2 Solutions with Full Steps

6.2.1 Q1

The answer is $\frac{10}{21}$.

There are $9 \times 8 \times 7 = 504$ ways to choose three distinct integers between 1 and 9 inclusively. Suppose the three integers are $a$, $b$ and $c$. Then $a + b + c$ is odd if and only if:

- $a$, $b$ and $c$ are all odd, or
- $a$ is odd but $b$ and $c$ are even, or
- $b$ is odd but $c$ and $a$ are even, or
- $c$ is odd but $a$ and $b$ are even

For the first case, there are $5 \times 4 \times 3 = 60$ ways to choose such a number. For the rest, there are $5 \times 4 \times 3 = 60$ ways each. Therefore the total number of ways to choose three distinct integers such that the sum is odd is $60 + 60 \times 3 = 240$.

Thus, the probability is $\frac{240}{504} = \frac{10}{21}$.

6.2.2 Q2

The answers are $(0, 6), (0, -6), (8, 10), (-8, 10), (8, -10)$ and $(-8, -10)$

**Solution 1:**

Obviously, $x = 0$ and $y = \pm 6$ is one of the solutions. Moreover, since $(6, 8, 10)$ is a Pythagorean Triple, $x = \pm 8$ and $y = \pm 10$ are also solutions.

**Solution 2:**

\[
\begin{align*}
6^2 + x^2 &= y^2 \\
36 &= y^2 - x^2 \\
36 &= (y + x)(y - x)
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
y + x &= \pm d \\
y - x &= \frac{36}{\pm d}
\end{cases}
\end{align*}
\]

Here $d = 1, 2, 3, 4, 6, 9, 12, 18, 36$ are divisors of 36. By trial and error, we found $(x, y) = (0, 6), (0, -6), (8, 10), (-8, 10), (8, -10)$ or $(-8, -10)$. 

21
6.2.3 Q3

The answers are (0, 1, 2) and (−2, −3, −4).

\[
\begin{cases}
xy + x + y = 1 \\
yz + y + z = 5 \\
zx + z + x = 2 \\
\end{cases}
\]

\[
\begin{cases}
(x + 1)(y + 1) = 2 \\
(y + 1)(z + 1) = 6 \\
(z + 1)(x + 1) = 3 \\
\end{cases}
\]

Therefore

\[
x = x + 1 - 1 = \sqrt{(x + 1)^2} - 1 = \sqrt{(x + 1)(y + 1) \cdot (z + 1)(x + 1)} - 1 = \sqrt{\frac{2 \times 3}{6}} - 1 = ±1 - 1 = 0 \text{ or } -2
\]

Similarly, \(y = 1\) or \(-3\) and \(z = 2\) or \(-4\).

6.2.4 Q4

The answer is 2401.

To form a triangle, \(a\) and \(b\) must satisfy the following inequalities:

\[
\begin{cases}
a + b > 100 \\
a < b \\
a, b < 100 \\
a, b \text{ are integers}
\end{cases}
\]

This corresponding to all lattice points inside the triangle with vertices (0, 100), (50, 50) and (100, 100). The number of lattice points of the triangle including the boundary is \(\frac{(100−0+1) \cdot (1) \cdot (100−50+1)}{2} = 2601\). The number of lattice points on the boundary is \((100 − 0 + 1) + (50 − 0 + 1) + (100 − 50 + 1) − 3 = 200\). Therefore the number of required lattice points is \(2601 − 200 = 2401\).
6.2.5 Q5

The answer is \( \frac{864}{25} \). 34.14 and 34.56 is also acceptable.

Assume the area of \( \triangle AXB \) is \( w \), \( \triangle BXC \) is \( x \), \( \triangle CXD \) is \( y \) and \( \triangle DXA \) is \( z \). Then \( x^2 - z^2 = wy \) since \( ABCD \) is a trapezoid. Moreover,

\[
\begin{align*}
x + w &= \text{Area of } \triangle ABC = \frac{9 \times 12}{2} = 54 \\
x + y &= \text{Area of } \triangle BCD = \frac{12 \times 16}{2} = 96
\end{align*}
\]

But \( y = \frac{x^2}{w} \). Therefore:

\[
\begin{align*}
w &= 54 - x \\
x^2 &= 96 - x
\end{align*}
\]

Multiplying the two equations gives:

\[
x^2 = (54 - x)(96 - x)
\]

\[
x^2 = x^2 - 150x + 5184
\]

\[
x = \frac{5184}{150}
\]

\[
x = \frac{864}{25}
\]

6.2.6 Q6

The answer is 886.

If \( 4^{999} + 4^{1111} + 4^x \) is a perfect square, it would be of the form \( (a + b)^2 \) for positive integers \( a \) and \( b \). As \( (a + b)^2 = a^2 + 2ab + b^2 \), we have either

\[
\begin{align*}
a^2 &= 4^{999} \\
2ab &= 4^{1111} \\
b^2 &= 4^x
\end{align*}
\]

Or

\[
\begin{align*}
a^2 &= 4^{999} \\
2ab &= 4^x \\
b^2 &= 4^{1111}
\end{align*}
\]

Or

\[
\begin{align*}
a^2 &= 4^{1111} \\
2ab &= 4^{999} \\
b^2 &= 4^x
\end{align*}
\]
For the first set of equations, we found \( b = 2^{1222} \), which means \( x = 1222 \). For the second set of equations, we found \( 2ab = 2^{2111} \), which means \( x = \frac{2111}{2} \). But this is not an integer and should be rejected. For the third set of equations, we found \( b = 2^{886} \), which implies \( x = 886 \). Out of the two, 886 is the minimum, hence it is the answer.

\[ \text{6.2.7 Q7} \]

The answer is 18.

The sum and product of roots are:

\[
\begin{align*}
\alpha + \beta &= 2 - k \\
\alpha \beta &= k^2 + 3k + 5
\end{align*}
\]

Therefore:

\[
\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \beta
\]

\[
= (2 - k)^2 - 2 (k^2 + 3k + 5)
\]

\[
= -k^2 - 10k - 6
\]

Since \( \alpha \) and \( \beta \) are real, the discriminant of the quadratic equation must be nonnegative, i.e.,

\[
\Delta \geq 0
\]

\[
(2 - k)^2 - 4 (k^2 + 3k + 5) \geq 0
\]

\[
-3k^2 - 16k - 16 \geq 0
\]

\[
3k^2 + 16k + 16 \leq 0
\]

\[
3(k + 4)(k + \frac{4}{3}) \leq 0
\]

\[
-4 \leq k \leq -\frac{4}{3}
\]

By substituting the endpoints of the feasible interval of \( k \) into \( \alpha^2 + \beta^2 \), we know its maximum is 18 when \( k = -4 \).
6.2.8 Q8

The answer is \( \frac{1}{53} \)

**Solution 1:**
Solving \( x + \frac{1}{x} = 4 \) gives \( x = 2 \pm \sqrt{3} \). Hence

\[
x^3 = (2 \pm \sqrt{3})^3
\]
\[
= 8 \pm 3(4 \sqrt{27}) + 3(2) \sqrt{9} \pm \sqrt{3}
\]
\[
= 26 \pm 15 \sqrt{3}
\]

Therefore

\[
\frac{x^3}{x^6 + x^3 + 1} = \frac{26 \pm 15 \sqrt{3}}{(26 \pm 15 \sqrt{3})^2 + 26 \pm 15 \sqrt{3} + 1}
\]
\[
= \frac{676 \pm 2(15)(26) \sqrt{3} + 225(3) + 27 \pm 15 \sqrt{3}}{26 \pm 15 \sqrt{3}}
\]
\[
= \frac{1378 \pm 795 \sqrt{3}}{1378^2 - 795^2(3)}
\]
\[
= \frac{35828 \pm 20670 \sqrt{3} + 20670 \sqrt{3} - 11925(3)}{2809}
\]
\[
= \frac{53}{2809}
\]
\[
= \frac{1}{53}
\]

**Solution 2:**
Define \( r_n = x^n + \frac{1}{x^n} \). Therefore \( r_0 = 2 \) and \( r_1 = 4 \) and

\[
r_2 = r_1 r_1 - r_0 = 14
\]
\[
r_3 = r_1 r_2 - r_1 = 52
\]

Thus

\[
\frac{x^3}{x^6 + x^3 + 1} = \left( \frac{x^6 + x^3 + 1}{x^3} \right)^{-1}
\]
\[
= \left( x^3 + \frac{1}{x^3} + 1 \right)^{-1}
\]
\[
= (r_3 + 1)^{-1}
\]
\[
= \frac{1}{53}
\]
6.2.9 Q9

The answer is 1

The problem is equivalent to finding \( \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{n^m} : \)

\[
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{n^m} = \sum_{n=2}^{\infty} \frac{1}{n^{n-1}} \\
= \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \\
= \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) \\
= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots \\
= 1
\]

6.2.10 Q10

The answer is \(-8\).

\[
3x^2 + 2y^2 + 2z^2 + 4xy + 2yz + 2zx - 4x - 4y - 8z \\
= 2(x + y + z)^2 - 4(x + y + z) + x^2 - 4z - 2yz - 2zx \\
= 2(2)^2 - 4(2) + x^2 - 2z(2 + y + x) \\
= x^2 - 2z(2 + 2 - x - z + x) \\
\geq 2z^2 - 8z \\
= 2(z - 2)^2 - 8 \\
\geq -8
\]

Equality occurs when \( x = y = 0, z = 2 \). Hence the minimum is \(-8\).
6.2.11 Q11

The answer is $\sqrt{1 + 16\pi^2} - \frac{1}{2} \sqrt{1 + 4\pi^2} + \frac{\sinh^{-1} 4\pi - \sinh^{-1} 2\pi}{4\pi}$. Any other equivalent form is acceptable.

The curve is $r = 1 + \frac{\theta}{2\pi}$. Therefore $\frac{dr}{d\theta} = \frac{1}{2\pi}$. Hence the arc length is:

$$
\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{\left(\frac{2\pi + \theta}{4\pi^2}\right)^2 + \frac{1}{4\pi^2}} \, d\theta
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(2\pi + \theta)^2 + 1} \, d\theta
$$

Substitute $x = 2\pi + \theta$ and then $x = \sinh y$. Thus:

$$
\frac{1}{2\pi} \int_0^{2\pi} \sqrt{(2\pi + \theta)^2 + 1} \, d\theta = \frac{1}{2\pi} \int_{\sinh^{-1} 4\pi}^{\sinh^{-1} 12\pi} \cosh^2 y \, dy
$$

$$
= \frac{1}{2\pi} \cdot \frac{y + \sinh y \cosh y}{2} \bigg|_{y=\sinh^{-1} 4\pi}^{y=\sinh^{-1} 12\pi}
$$

$$
= \frac{1}{2\pi} \left( \frac{\sinh^{-1} 4\pi + 4\pi \sqrt{16\pi^2 + 1}}{2} - \frac{\sinh^{-1} 2\pi + 2\pi \sqrt{4\pi^2 + 1}}{2} \right)
$$

$$
= \sqrt{16\pi^2 + 1} - \frac{1}{2} \sqrt{4\pi^2 + 1} + \frac{\sinh^{-1} 4\pi - \sinh^{-1} 2\pi}{4\pi}
$$

6.2.12 Q12

The answer is $\frac{14\pi}{3}$.

The area is:

$$
\int_0^{2\pi} r^2 \, d\theta = \int_0^{2\pi} \left(1 + \frac{\theta}{2\pi}\right)^2 \, d\theta
$$

$$
= \int_0^{2\pi} \left(1 + \frac{\theta}{\pi} + \frac{\theta^2}{4\pi^2}\right) \, d\theta
$$

$$
= \theta + \frac{\theta^2}{2\pi} + \frac{\theta^3}{12\pi^2}\bigg|_{\theta=0}^{2\pi}
$$

$$
= 2\pi + \frac{4\pi^2}{2\pi} + \frac{8\pi^3}{12\pi^2}
$$

$$
= \frac{14\pi}{3}
$$
Chapter 7

Secondary Three, Solutions

7.1 Outline of Solutions

1. 3
2. 180
3. $\frac{1}{15}$
4. 2475
5. 246
6. 2
7. 243
8. 4 : 9
9. 4
10. $\frac{\sqrt{30} + 6\sqrt{5} - \sqrt{5} + 1}{8}$
7.2 Solutions with Full Steps

7.2.1 Q1

The answer is 3.

\[
\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+11)(x+12)} + \frac{1}{x+12} = \frac{1}{4}
\]

\[
\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+2} - \frac{1}{x+3} + \ldots + \frac{1}{x+11} - \frac{1}{x+12} + \frac{1}{x+12} = \frac{1}{4}
\]

\[
\frac{1}{x+1} = \frac{1}{4}
\]

\[
x = 3
\]

7.2.2 Q2

The answer is 180.

If a number is a perfect cube, its prime factorization will be of the form

\[2^{3p} \cdot 3^{3q} \cdot 5^{3r} \cdot 7^{3s} \ldots\]

for nonnegative integers \(p, q, r, \ldots\). Therefore,

\[150b = 2^{3p} \cdot 3^{3q} \cdot 5^{3r} \cdot 7^{3s} \ldots\]

\[2 \cdot 3 \cdot 5^2 b = 2^{3p} \cdot 3^{3q} \cdot 5^{3r} \cdot 7^{3s} \ldots\]

\[b = 2^{3p-1} \cdot 3^{3q-1} \cdot 5^{3r-2} \cdot 7^{3s} \cdot 11^{3t} \ldots\]

\(b\) will be minimum when \(p, q, r = 1\) and \(s, t, \ldots = 0\), i.e., \(b = 2^2 \cdot 3^2 \cdot 5 = 180\).
The answer is $\frac{1}{15}$.

The factorizations of integers from 1 to 10 are:

\[
\begin{align*}
1 &= 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 \\
2 &= 2^1 \cdot 3^0 \cdot 5^0 \cdot 7^0 \\
3 &= 2^0 \cdot 3^1 \cdot 5^0 \cdot 7^0 \\
4 &= 2^2 \cdot 3^0 \cdot 5^0 \cdot 7^0 \\
5 &= 2^0 \cdot 3^0 \cdot 5^1 \cdot 7^0 \\
6 &= 2^1 \cdot 3^1 \cdot 5^0 \cdot 7^0 \\
7 &= 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^1 \\
8 &= 2^3 \cdot 3^0 \cdot 5^0 \cdot 7^0 \\
9 &= 2^0 \cdot 3^2 \cdot 5^0 \cdot 7^0 \\
10 &= 2^1 \cdot 3^0 \cdot 5^1 \cdot 7^0
\end{align*}
\]

A number is perfect square if and only if the index of each prime in the factorization is even.

For 1 to 10, the highest index is 6 for 2, 4 for 3, 2 for 5 and 1 for 7. Therefore we can have $4 \times 3 \times 2 \times 1 = 24$ distinct perfect square results. But resulting as the square of 1 or prime (2, 3, 5) is not allowed, since the three numbers are required as different. Also, $2^x \cdot 3^y \cdot 5^z (y \neq 0)$ and $2^x \cdot 3^4 \cdot 5^z$ is impossible. The remaining cases are:

<table>
<thead>
<tr>
<th>Factorization</th>
<th>Equivalent Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^2 \cdot 3^2 \cdot 5^0$</td>
<td>$1 \times 4 \times 9$, $2 \times 3 \times 6$</td>
</tr>
<tr>
<td>$2^2 \cdot 3^0 \cdot 5^2$</td>
<td>$2 \times 5 \times 10$</td>
</tr>
<tr>
<td>$2^4 \cdot 3^0 \cdot 5^0$</td>
<td>$1 \times 2 \times 8$</td>
</tr>
<tr>
<td>$2^4 \cdot 3^2 \cdot 5^0$</td>
<td>$2 \times 8 \times 9$, $3 \times 6 \times 8$</td>
</tr>
<tr>
<td>$2^4 \cdot 3^0 \cdot 5^2$</td>
<td>$5 \times 8 \times 10$</td>
</tr>
<tr>
<td>$2^6 \cdot 3^0 \cdot 5^0$</td>
<td>$2 \times 4 \times 8$</td>
</tr>
<tr>
<td>$2^6 \cdot 3^2 \cdot 5^0$</td>
<td>$-$</td>
</tr>
<tr>
<td>$2^6 \cdot 3^0 \cdot 5^2$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

There are 8 different combinations in total. Counting order, the total number of ways to choose those three numbers is $8 \times 3! = 48$. Also, there are $10 \times 9 \times 8 = 720$ ways to pick three distinct numbers. Hence the probability is $\frac{48}{720} = \frac{1}{15}$. 

30
7.2.4 Q4

The answer is 2475.

\[
\frac{1}{2} + \frac{1}{3} + \frac{2}{3} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \ldots + \frac{1}{100} + \frac{2}{100} + \ldots + \frac{99}{100}
\]

\[
= \left( \frac{1}{2} \right) + \left( \frac{1}{3} + \frac{2}{3} \right) + \left( \frac{1}{4} + \frac{2}{4} + \frac{3}{4} \right) + \ldots + \left( \frac{1}{100} + \ldots + \frac{99}{100} \right)
\]

\[
= \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \ldots + \frac{99}{2}
\]

\[
= \frac{1}{2}(99) \left( \frac{1}{2} + \frac{99}{2} \right)
\]

\[
= 2475
\]

7.2.5 Q5

The answer is 246.

(The full-step solution is working out)

7.2.6 Q6

The answer is 2.

Note that Euler’s Theorem works as long as the totient function of the modulus is not 0. We first compute the (nested) Euler’s totient function of 100 first:

\[
\phi^0(100) = 100
\]

\[
\phi^1(100) = 40
\]

\[
\phi^2(100) = 16
\]

\[
\phi^3(100) = 8
\]

\[
\phi^4(100) = 4
\]

\[
\phi^5(100) = 2
\]
Therefore:

\[
2^{3125} \equiv 2 \pmod{\phi^1} \\
\equiv 2^{\phi^1} \\
\equiv 2 \pmod{\phi^2} \\
\equiv 2^{\phi^2} \\
\equiv 2 \pmod{\phi^3} \\
\equiv 2^{\phi^3} \\
\equiv 2 \pmod{\phi^4} \\
\equiv 2^{\phi^4} \\
\equiv 2 \pmod{\phi^5} \\
\equiv 2^{\phi^5} \\
\equiv 2 
\]

That means the remainder is 2.

7.2.7 Q7

The answer is 243.

Write 0 to 9 in three columns. We can choose three numbers from these columns to form a three-digit number.

If the decimal representation of a three-digit number contains exactly one “2”, then one column must be “2”, and the other two can be anything except “2”. Hence there are \(3 \times (9 \times 9) = 243\) such numbers.
The answer is 4 : 9. Do not accept non-integral ratios such as 1 : 2.25, since this is not the simplest form.

**Solution 1:**
Notice that $AN$ is isogonal to $AM$ in $\triangle ABC$. If the angle bisector of $\angle BAC$ intersect $BC$ at $K$, then

\[
\frac{BN}{NC} \cdot \frac{BM}{MC} = \left( \frac{BK}{KC} \right)^2
\]

By angle bisector theorem, $\frac{BK}{KC} = \frac{2}{3}$. Therefore

\[
\frac{BN}{NC} \cdot 1 = \left( \frac{2}{3} \right)^2
\]


**Solution 2:**
If the ratio is fixed, then it should be also correct for a specific $\triangle ABC$. Now suppose all the system lies on a coordinates plane with $A = (0, 0), B = (0, 2)$ and $C = (3, 0)$. Thus $M = \left(1, \frac{3}{2}\right)$. The line $AN$ can be found by reflecting $AM$ along the angle bisector of $\angle BAC$, i.e., $y = x$. Suppose $M'$ is the image of $M$ after reflection. Then $M' = \left(\frac{3}{2}, 1\right)$. That means the line $AN$ is $2y = 3x$. By intercept form, the equation of $BC$ is $\frac{x}{3} + \frac{y}{2} = 1$. If $N = (\xi, \eta)$, then

\[
\begin{cases} 
2\eta = 3\xi \\
\frac{\xi}{3} + \frac{\eta}{2} = 1 
\end{cases}
\]

Therefore the $y$-coordinates of $N$ is $\frac{18}{13}$. Thus

\[
BN : NC = \left( 2 - \frac{18}{13} \right) : \left( \frac{18}{13} - 0 \right) = 8 : 18 = 4 : 9
\]
7.2.9  Q9

The answer is 4.

Solution 1:
Let \( K = \sum_{k=0}^{\infty} \frac{F_k}{2^k} \). Then by \( F_n = F_{n-1} + F_{n-2} \), we have:
\[
K + \frac{K}{2} = 2K - 2
\]
\[
K = 4
\]

Solution 2:
\[
\sum_{k=0}^{\infty} \frac{F_k}{2^k} = \frac{1}{1 - \frac{1}{2} - \left(\frac{1}{2}\right)^2} = 4
\]

7.2.10  Q10

The answer is \( \frac{\sqrt{30 + 6\sqrt{5}} - \sqrt{5} + 1}{8} \). Any equivalent form is acceptable.

Since \( \sin 60^\circ = \frac{\sqrt{3}}{2} \), \( \cos 60^\circ = \frac{1}{2} \), \( \sin 18^\circ = \frac{\sqrt{5} - 1}{4} \) and \( \cos 18^\circ = \sqrt{1 - \sin^2 18^\circ} \)
\[
\cos 18^\circ = \sqrt{1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2}
\]
\[
= \sqrt{16 - 5 - 1 + 2\sqrt{5}}
\]
\[
= \frac{\sqrt{10 + 2\sqrt{5}}}{4}
\]

Therefore
\[
\sin 42^\circ = \sin (60^\circ - 18^\circ)
\]
\[
= \sin 60^\circ \cos 18^\circ - \cos 60^\circ \sin 18^\circ
\]
\[
= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{10 + 2\sqrt{5}}}{4} - \frac{1}{2} \cdot \frac{\sqrt{5} - 1}{4}
\]
\[
= \frac{\sqrt{30 + 6\sqrt{5}} - \sqrt{5} + 1}{8}
\]
Chapter 8

Secondary Four, Solutions

8.1 Outline of Solutions

1. $180^\circ$
2. $\frac{1}{5}$
3. $\frac{4\pi}{9}$
4. 86
5. 886
6. $(2, 3), (-2, 3), (2, -3), (-2, -3), (4, 3), (-4, 3), (4, -3), (-4, -3)$
7. 8
8. $\frac{15}{32}$
9. $\frac{2\pi}{3}$
10. $\frac{\pi}{2} + i \cosh^{-1} 2$
11. $-\infty$
8.2 Solutions with Full Steps

8.2.1 Q1

The answer is $180^\circ$. $\pi$ is also acceptable.

**Solution 1:**
Since the figure as 7 vertices and every edge “jumps” through 2 vertices, the sum of interior angles is $180^\circ(7 - 2 \times (2 + 1)) = 180^\circ$

**Solution 2:**
Without loss of generality, we may assume that the figure is “regular”. Therefore, all the marked angles are the same. Refering to the figure 8.1, $\angle RQS = \angle PQM = \frac{180^\circ(7-2)}{7} = \frac{900^\circ}{7}$ since it is an interior angle of a regular heptagon (7-gon). Therefore:

$$\angle QSA = \angle QRA = 180^\circ - \angle QRP$$
$$= 2\angle PQR$$
$$= 2(180^\circ - \angle PQM)$$
$$= 360^\circ - \frac{1800^\circ}{7}$$
$$= \frac{720^\circ}{7}$$

Hence, in quadrilateral $ARQS$, $\angle SAR = 360^\circ - \angle QSA - \angle QRA - \angle RQS = 360^\circ - \frac{900^\circ}{7} - \frac{720^\circ}{7} - \frac{720^\circ}{7} = \frac{180^\circ}{7}$. Multiply this by 7 will get the total, i.e., $180^\circ$.

8.2.2 Q2

The answer is $\frac{1}{5}$. 0.2 is also acceptable.

$$\log_{1440} 2 + \log_{1440} \sqrt[5]{9} + \log_{1440} \sqrt[5]{5} = \log_{1440} \left(2 \cdot \sqrt[5]{9} \cdot \sqrt[5]{5}\right)$$
$$= \log_{1440} \left(\sqrt[5]{2^5 \cdot 9 \cdot 5}\right)$$
$$= \log_{1440} \left(\sqrt[5]{1440}\right)$$
$$= \frac{1}{5}$$
Figure 8.1: Figure to MHAS 2004/4 Q1 (Solution)

Figure 8.2: Figure to MHAS 2004/4 Q3 (Solution)
8.2.3 Q3

The answer is $\frac{4\pi}{9}$.

**Solution 1:**
Construct $G$ as the intersection of $AC$ and circle $DEF$, and $O$ as the center of $DEF$ as shown in figure 8.2. Let $r = OD$ be the radius of the circle. Then $\triangle ODG, \triangle OGF$ and $\triangle CGF$ are three congruent equilateral triangles. Hence $CD = 2CF = 2r$. But $CD + CF = 2$. Therefore $r = \frac{2}{3}$. Thus the area of the circle is $\pi \left(\frac{2}{3}\right)^2 = \frac{4\pi}{9}$.

**Solution 2:**
As shown in figure 8.2, Let $x = AD = BE = CF$, $y = AE = BF = CD$, $h = DE = EF = FD$ and $r = OD = OE = OF$. Since $\triangle CFD$ is a right triangle with $\angle DCF = 60^\circ$ and $\triangle ODF$ is a isosceles triangle with $\angle DOF = 120^\circ$ we have:

\[
\begin{align*}
x + y &= 2 \quad \text{(Given)} \\
h^2 &= y^2 - x^2 \quad \text{(Pythagoras’ Theorem)} \\
h &= \frac{y}{x} = \tan 60^\circ = \sqrt{3} \\
h^2 &= r^2 + r^2 - 2rr \cos 120^\circ \quad \text{(Cosine Law)}
\end{align*}
\]

\[
\begin{align*}
h &= \sqrt{2(y - x)} \\
h &= x\sqrt{3} \\
h &= r\sqrt{3}
\end{align*}
\]

\[
\begin{align*}
3x^2 &= 2(2 - x) - 2x \\
x &= r
\end{align*}
\]

\[
\Rightarrow r = \frac{2}{3} \quad \text{or} \quad -2 \quad \text{(Rejected)}
\]

Therefore, the area of the circle is $\pi r^2 = \frac{4\pi}{9}$.

8.2.4 Q4

The answer is 86.

Since all numbers are relatively prime, we can apply Euler’s Theorem straight
forwardly. We first compute the (nested) Euler’s totient function of 101 first:

\[
\begin{align*}
\phi^0(101) &= 101 \\
\phi^1(101) &= 100 \\
\phi^2(101) &= 40 \\
\phi^3(101) &= 16 \\
\phi^4(101) &= 8 \\
\phi^5(101) &= 4 \\
\phi^6(101) &= 2 \\
\phi^7(101) &= 1
\end{align*}
\]

Therefore:

\[
\begin{align*}
&2 \equiv 2 \pmod{\phi^7} \\
&\equiv 2^{101} \equiv 2 \pmod{\phi^6} \\
&\equiv 2^{40} \equiv 2 \pmod{\phi^5} \\
&\equiv 2^{16} \equiv 2 \pmod{\phi^4} \\
&\equiv 2^{8} \equiv 2 \pmod{\phi^3} \\
&\equiv 2^{4} \equiv 2 \pmod{\phi^2} \\
&\equiv 2^{2} \equiv 2 \pmod{\phi^1} \\
&\equiv 2 \times (2^7)^6 \equiv 2 \times 27^6 \equiv 2 \times 729^3 \equiv 2 \times 22^3 \equiv 21296 \equiv 86 \pmod{\phi^0}
\end{align*}
\]

That means the remainder is 86.
8.2.5 Q5

The answer is 886.

(Please read solution to Q6 of secondary 2.)

8.2.6 Q6

The answers are (2, 3), (−2, 3), (2, −3), (−2, −3), (4, 3), (−4, 3), (4, −3), (−4, −3).

The problem can be restated as “Find all integral solutions to $x^4 - y^4 - 20x^2 + 28y^2 - 107 = 0$.” If we can transform $x^4 - y^4 - 20x^2 + 28y^2 - 107$ into form $(x^2 + y^2 + a)(x^2 - y^2 + b) - c$, we can find $x$ and $y$ using the prime factorization of $c$.

Now expand $(x^2 + y^2 + a)(x^2 - y^2 + b) - c$ to give $x^4 - y^4 + (b + a)x^2 + (b - a)y^2 + (ab - c)$. By comparing coefficient with $x^4 - y^4 - 20x^2 + 28y^2 - 107$, we have:

$$b + a = -20$$
$$b - a = 28$$
$$ab - c = -107$$

Therefore $a = -24, b = 4, c = 11$. That means $x$ and $y$ satisfies the relationship $(x^2 + y^2 - 24)(x^2 - y^2 + 4) = 11$. Thus

$$\begin{cases} x^2 + y^2 - 24 = 11 \\ x^2 - y^2 + 4 = 1 \end{cases}$$

or

$$\begin{cases} x^2 + y^2 - 24 = -11 \\ x^2 - y^2 + 4 = -1 \end{cases}$$

Clearly, the first set of equations should be rejected because of $x^2 + y^2 = -25$. The second set will give (4, 3), (4, −3), (−4, 3), (−4, −3) as solutions. The third set will give (2, 3), (2, −3), (−2, 3), (−2, −3) as solutions. The fourth set will not give integral solutions, and hence rejected.
8.2.7 Q7

The answer is 8.

Since
\[ \cos 15^\circ = \cos (45^\circ - 30^\circ) \]
\[ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \]
\[ = \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \frac{1}{2} \]
\[ = \frac{\sqrt{6} + \sqrt{2}}{2} \]

Therefore \( a = 6 \) and \( b = 2 \), i.e., \( a + b = 8 \).

8.2.8 Q8

The answer is \( \frac{15}{32} \). 0.46875 is also acceptable.

**Solution 1:**

\[
\frac{1}{5} + \frac{4}{5^2} + \frac{9}{5^3} + \frac{16}{5^4} + \ldots \\
= \frac{1}{5} \left( 1 + \frac{1}{5} + \frac{1}{5^2} + \ldots \right) + \frac{3}{5^2} \left( 1 + \frac{1}{5} + \frac{1}{5^2} + \ldots \right) + \frac{5}{5^3} \left( 1 + \frac{1}{5} + \ldots \right) + \ldots \\
= \frac{1}{5} \left( \frac{1}{1 - \frac{1}{5}} \right) + \frac{3}{5^2} \left( \frac{1}{1 - \frac{1}{5}} \right) + \frac{5}{5^3} \left( \frac{1}{1 - \frac{1}{5}} \right) + \ldots \\
= \frac{5}{4} \left( \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} + \ldots \right) \\
= \frac{5}{4} \left( 2 \left( \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots \right) - \left( \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots \right) \right) \\
= \frac{5}{4} \left( 2 \cdot \frac{1}{\left( \frac{1}{5} - 1 \right)^2} - \frac{1}{1 - \frac{1}{5}} \right) \\
= \frac{5}{4} \left( \frac{25}{16} - \frac{1}{4} \right) \\
= \frac{15}{32}
\]
Solution 2:

\[
\begin{align*}
&\frac{1}{5} + \frac{4}{5^2} + \frac{9}{5^3} + \frac{16}{5^4} + \ldots \\
&= \sum_{k=1}^{\infty} \frac{k^2}{5^k} \\
&= -\frac{1}{5} \left(1 + \frac{1}{5}\right) \\
&= -\frac{1}{5} \cdot \frac{6}{5} \\
&= -\frac{6}{25} \\
&= \frac{15}{32}
\end{align*}
\]

Solution 3:

\[
\begin{align*}
&\frac{1}{5} + \frac{4}{5^2} + \frac{9}{5^3} + \frac{16}{5^4} + \ldots \\
&= \sum_{k=1}^{\infty} \frac{k^2}{5^k} \\
&= \sum_{k=2}^{\infty} \frac{d^2}{dx^2} x^k - 3 \sum_{k=1}^{\infty} \frac{d}{dx} x^k + \sum_{k=0}^{\infty} x^k \\
&= \left. \frac{d^2}{dx^2} \frac{x^2}{1-x} - 3 \frac{d}{dx} \frac{x}{1-x} + \frac{1}{1-x} \right|_{x=\frac{1}{5}} \\
&= \left. \frac{2x^2}{(1-x)^3} + \frac{4x}{(1-x)^2} + \frac{2}{1-x} - \frac{3x}{(1-x)^2} - \frac{1}{1-x} + \frac{1}{1-x} \right|_{x=\frac{1}{5}} \\
&= \frac{2}{(1-\frac{1}{5})^3} + \frac{\frac{1}{5}}{(1-\frac{1}{5})^2} \\
&= \frac{15}{32}
\end{align*}
\]

8.2.9 Q9

The answer is \(\frac{2\pi}{3}\).

**Solution 1:**
The region of integration is a unit circle, and the integrand is describing the surface of a unit hemisphere. Therefore, the result of integral is equivalent to half the volume of a unit sphere, which is \(\frac{1}{2} \cdot \frac{4\pi}{3} = \frac{2\pi}{3}\).

**Solution 2:**
By transforming the coordinates space of the integrand from cartesian into polar,
we have
\[
\int \int_A \sqrt{1-x^2-y^2} \, dx \, dy = \int_0^1 \int_0^{2\pi} r \sqrt{1-(r \cos \theta)^2 - (r \sin \theta)^2} \, d\theta \, dr \\
= \int_0^1 \int_0^{2\pi} r \sqrt{1-r^2} \, d\theta \, dr \\
= \int_0^1 2\pi \sqrt{1-r^2} \, dr \\
= -\pi \int_1^0 \sqrt{k} \, dk \\
= \pi \left( \frac{k^{3/2}}{3/2} \right)_{k=0}^{k=1} \\
= \frac{2\pi}{3}
\]

8.2.10 Q10

The answer is \( \frac{\pi}{2} + i \cosh^{-1} 2 \). \( \frac{\pi}{2} + i \ln (2 + \sqrt{3}) \) or any other suitable values are also acceptable.

Assume \( z = a + bi \). Then \( \sin z = \sin a \cosh b + i \cos a \sinh b \). By comparing real and imaginary parts, we have:

\[
\begin{cases}
\sin a \cosh b = 2 \\
\cos a \sinh b = 0
\end{cases}
\]

Which means \( \cos a = 0 \), i.e., \( a = \pi \left(n + \frac{1}{2}\right) \) for any integer \( n \). Therefore \( b = \cosh^{-1} 2 \). Thus \( z = \pi \left(n + \frac{1}{2}\right) + \cosh^{-1} 2 \). In particular, \( \frac{\pi}{2} + i \cosh^{-1} 2 \) is the “simplest” solution.
8.2.11 Q11

The answer is $-\infty$.

$$0 + 1 - 2 \times 3 \div 4 + 5 - 6 \times 7 \div 8 + 9 - 10 \times \ldots$$

$$= \sum_{k=1}^{\infty} \left( k - \frac{(k+1)(k+2)}{k+3} \right)$$

$$= \sum_{k=1}^{\infty} \frac{k^2 + 3k - k^2 - 3k - 2}{k+3}$$

$$= -\sum_{k=1}^{\infty} \frac{2}{k+3}$$

$$= -\frac{2}{4} - \frac{2}{8} - \frac{2}{12} - \frac{2}{16} - \ldots$$

$$= -\frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \right)$$

$$= -\frac{1}{2} (\infty - 1)$$

$$= -\infty$$
Chapter 9

Statistical Information

Here are some statistical information about the competition:

<table>
<thead>
<tr>
<th>Item</th>
<th>Overall</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enrollments</td>
<td>46</td>
<td>31</td>
<td>5</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Contestants</td>
<td>34</td>
<td>15</td>
<td>5</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Average Score</td>
<td>16.8015</td>
<td>25.4667</td>
<td>6.9</td>
<td>11.4286</td>
<td>10.6786</td>
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<tr>
<td>Median Score</td>
<td>18.75</td>
<td>26</td>
<td>1</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>Maximum Score</td>
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<td>46</td>
<td>18.5</td>
<td>36</td>
<td>22.25</td>
</tr>
<tr>
<td>Minimum Score</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mean Deviation</td>
<td>10.0307</td>
<td>6.9689</td>
<td>7.48</td>
<td>11.3469</td>
<td>7.2755</td>
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<tr>
<td>Contestants Passing 30%</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Average Attempted Problems</td>
<td>7.9412</td>
<td>9.4</td>
<td>5</td>
<td>8</td>
<td>7.5714</td>
</tr>
<tr>
<td>Average Solved Problems</td>
<td>2.0821</td>
<td>2.7</td>
<td>0.6333</td>
<td>1.1429</td>
<td>1.0893</td>
</tr>
<tr>
<td>Part D Solvers</td>
<td>0</td>
<td>–</td>
<td>0</td>
<td>–</td>
<td>0</td>
</tr>
</tbody>
</table>
Chapter 10

Nominated Contestants

10.1 Secondary One

1. 劉子斌 (1B) – 46 marks
2. 柯俊宇 (1A) – 39 marks
3. 顏正綱 (1B) – 32 marks

10.2 Secondary Two

1. 林家榮 (2A) – 18.5 marks
2. 謝曉晴 (2A) – 14 marks
3. 郭潔玲 (2A), 朱家謙 (2A) – 1 mark

10.3 Secondary Three

1. 郭子彪 (3A) – 36 marks
2. 梁錦程 (3B) – 22 marks
3. 廖國良 (3B) – 16 marks
10.4 Secondary Four

1. 曾自鳴 (4E) – 22.25 marks
2. 何淑嫺 (4D) – 21 marks
3. 樊兆聰 (4D) – 14.25 marks
Chapter 11

Comments

11.1 Secondary One

The competition date was on 28th October, 2004. It was supposed that the competition held at room 517 (General Laboratory), but an important meeting is being conducted during that period. Therefore the competition site was changed to room 416 (Physics Laboratory).

Performance of contestants:

1. **Fair.** Most contestants knew that $2^{2k} \equiv 6 \pmod{10}$. One contestant misunderstood that “closest to 1000” does not include 1000 and answered 996. This answer is given 3 marks only instead of 7.

2. **Good.** Almost all contestants can solve this problem.

3. **Bad.** None of the contestants can answer this correctly. The common answers are 0%, 50% and 33$\frac{1}{3}$%.

4. **Satisfactory.**

5. **Unsatisfactory.** Only one contestant can solve this problem. That means almost all of them answer without reviewing whether it was right or not.

6. **Bad.** Although the formulae of perimeter/area of square/circle had been taught in primary school, none of the contestant had answered this correctly. Some mysterious answers like 0.284, 0.785 and 0.125 appear several times. I suspect there is copying among the contestants.

7. **Fair.**
8. **Bad.** Nearly all contestants answered 0 or leaving this blank.

9. **Unsatisfactory.** Many contestants do not know about the term “integral solutions” and answered with a rational pair. Others answer without reviewing. None of the contestants can solve this problem.

10. **Good.** Almost all contestants can solve this problem.

### 11.2 Secondary Two

The competition date was on 11th November, 2004. It was originally planned to have the competition a week earlier, but an inter-form talk was suddenly held on that day and therefore the competition was postponed.

Performance of contestants:

1. **Bad.** Even though half of the contestants attempted this problem, they gave answer far less than the actual value. It seems that they are not very familiar with combinatorical problems.

2. **Satisfactory.** Almost all of them can find the solution pair $x = 8, y = 10$. However, the nonpositive solutions are are ignored.

3. **Unsatisfactory.**

4. **Bad.** It is obvious that the number of such triangles are more than 10 (consider “90-99-100” to “99-99-100”), but still quite a number of contestants answered it as numbers below 10.

5. **Unsatisfactory.**

6. **Satisfactory.** One contestant can almost get the solution (his is 887); but most of them answered 999.

7. **Unsatisfactory.**

8. **Unsatisfactory.**

9. **Good.**

10. **Unsatisfactory.**

11. —

12. —
11.3 Secondary Three

The competition date was on 5th November, 2004.

Performance of contestants:

1. **Satisfactory.** However, some of the contestants do not known the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and tried to expand all the denominators.

2. **Unsatisfactory.** Most of the contestants do not know how to relate this problem to prime factorization of of 150. One even can’t calculate $30^3$ correctly.

3. **Unsatisfactory.**

4. **Bad.** It seems that most contestants do not know how to group fractions of similar patterns, or adding things wrongly.

5. **Satisfactory.**

6. **Satisfactory.**

7. **Unsatisfactory.** No one could find a bijection from the original statement to the table as in (7.2.7).

8. **Unsatisfactory.**

9. **Fair.**

10. **Unsatisfactory.**

11.4 Secondary Four

The competition date was on 29th October, 2004. Because of time clash with the Halloween Party held at the same day, a contestant was unable to attend the competition. (She was responsible for the party.) She was arranged to attempt the paper two weeks later. Some contestants misunderstood that the competition was a group event. In fact, there is only individual events in MHAS. Group events will **never** be held in MHAS. It is found that the concept of logarithm had not been taught in S4 level, although it was in syllabus in the past. The basic definition, i.e., $\log_a b = n \Leftrightarrow a = b^n$ was introduced during the competition.

Performance of contestants:

1. **Good.** One contestant had carelessly written $180^\circ$C.
2. **Unsatisfactory.** Due to the lack of knowledge of logarithm, none of the contestants can solve this problem. One of the contestant could grasp the concept of logarithm, but he had used a false theorem.

3. **Unsatisfactory.** The answer 1.767 appears four times with unknown reason.

4. **Unsatisfactory.**

5. **Bad.** It appears that none of the contestants could relate this question to the expansion of \((a + b)^2\).

6. **Satisfactory.** Almost all contestants could find the pair \((2, 3)\). However, the requirement of the problem is *all solutions*, and there are totally eight. The contestants who answered \((2, 3)\) will be given 1.25 marks, i.e., \(\frac{5}{8}\) of the total marks of this question.

7. **Unsatisfactory.** Only one contestant could give the right answer. It appears that many contestant do not memorize the values of \(\sin 45^\circ\), \(\cos 30^\circ\), etc. and the identity \(\cos(a - b) = \cos a \cos b + \sin a \sin b\).

8. **Unsatisfactory.** Nearly all contestants answered \(\infty\).

9. —

10. —

11. **Bad.** Only one contestant could successfully transform the original question to \(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \ldots\), but he answered \(\infty\).