**Strategy in Algebra**

**Make Suitable Substitution**

In algebraic problems, we have to make some substitution to the variable to reduce the number of variable or just transform the problem into an easier one. There is no regulation for how to make a “good” substitution, but usually the “shape” of the formula will give you some clues – and this will be become easier to catch if you are experienced.

**Example A:** (HK Team Selection 2001 Q4) Let $a$, $b$ and $c$ be positive real numbers. Prove that \((a + b)^2 + (a + b + 4c)^2 \geq \frac{100abc}{a + b + c}\).

**Solution A:**

Observe that “$a + b$” appears on 3 places. Therefore, we can make a substitution $x = a + b$ to reduce the number of variables. Of course, we have to get rid of the “$ab$” first, but that is easy using the AM-GM inequality:

$$\sqrt{ab} \leq \frac{a + b}{2}$$

Thus the original problem becomes:

$$x^2 + (x + 4c)^2 \geq \frac{100c}{x + c} \left(\frac{x}{2}\right)^2$$

$$x^2 + (x + 4c)^2 \geq \frac{25x^2c}{x + c}$$

Now divide $c^2$ to both sides:

$$\left(\frac{x}{c}\right)^2 + \left(\frac{x + 4c}{c}\right)^2 \geq \frac{25\left(\frac{x}{c}\right)^2}{\frac{x}{c} + 1}$$

The value \(\frac{x}{c}\) appears on both sides, twice. Thus we may again substitute it as $y$:

$$y^2 + (y + 4)^2 \geq \frac{25y^2}{y + 1}$$

$$2y^3 - 15y^2 + 24y + 16 \geq 0$$

$$(y - 4)(2y + 1) \geq 0$$

The last statement is obviously true, and thus the original statement. Q.E.D.
Dealing with Infinite Expression

In mathematical contests solving expressions involving infinite number of terms is common. The technique is usually assuming the answer exists, and substitutes this back to the original statement.

**Example B:** (HKMO 1998/II Q5) Solve $2x + 3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$

**Solution B:**

If such an $x$ exists, then $\sqrt{2 + (2x + 3)} = \sqrt{2 + \left( \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}} \right)} = 2x + 3$. Thus, $2x + 5 = (2x + 3)^2$, i.e., $x = -2$ or $-\frac{1}{2}$. But when $x = -2$, LHS = $2x + 3 = -1 < 0$. But RHS is positive for obvious reason. Therefore, $x = -\frac{1}{2}$ is the solution.

**Example C:** Solve $x^{x^{x^{\ldots}}} = 2$, where we define $x^{x^{\ldots}} = x^{(x^{(x^{\ldots})})}$.

**Solution C:**

If such an $x$ exists, then $2 = x^{(x^{(x^{\ldots})})} = x^2$, i.e., $x = \sqrt{2}$.
Observing Invariants

Invariants are quantities that remain unchanged with specific transformations. These can be used to disprove many kinds of problems.

Example D: The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$:
1. Switch $a$ and $c$,
2. Replace $x$ by $x + t$, where $t$ is a real number.

By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

Solution D:
A property that is especially useful to quadratic polynomial is its discriminant $\Delta = b^2 - 4ac$. This value is invariant under step 1. For step 2,

$$a(x+t)^2 + b(x+t) + c = ax^2 + (2at + b)x + (at^2 + bt + c)$$

Therefore, the “new” $\Delta = (2at + b)^2 - 4a(at^2 + bt + c) = b^2 - 4ac$. Therefore $\Delta$ is invariant under both steps. Now since $\Delta$ of $x^2 - x - 2$ is 9 and $x^2 - x - 1$ is 5, the answer is no.

Example E: Given $x_0$ and $y_0$ such that $x_0 > y_0 > 0$. Define two sequences $x$ and $y$ such that $x_{n+1} = \frac{x_n + y_n}{2}$, $y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$. Find $x_\infty$ and $y_\infty$.

Solution E:
Observe that $x_{n+1}y_{n+1} = x_ny_n = x_0y_0$. Therefore, $x_ny_n$ is invariant. Assume $x_\infty$ and $y_\infty$ exists, then $x_\infty = \frac{x_\infty + y_\infty}{2}$ and $y_\infty = \frac{2x_\infty y_\infty}{x_\infty + y_\infty}$ (Note that $\infty + 1 = \infty$). So $x_\infty = y_\infty$. By the invariant we have $x_\infty = y_\infty = \sqrt{x_0y_0}$.
Important Numerical Values & Formulae

The following values must be memorized:
- \( \sqrt{2} \approx 1.414 \)
- \( \sqrt{3} \approx 1.732 \)
- \( \sqrt{5} \approx 2.236 \)
- \( \log 2 \approx 0.3010 \)

The following values should be noticed:
- \( \sqrt{7} \approx 2.646 \)
- \( \sqrt{6} = \sqrt{2} \cdot \sqrt{3} \approx 2.449 \)
- \( \sqrt{10} = \sqrt{2} \cdot \sqrt{5} \approx 3.162 \)
- \( \log 3 \approx 0.4771 \)
- \( \log 5 = 1 - \log 2 \approx 0.6990 \)
- \( \lg 10 = \frac{1}{\log 2} \approx 3.322 \)
- \( e \approx 2.718 \)

Important formulae that is not introduced in Elementary Algebra:
- \[ \sum_{k=1}^{n} k^2 = n(n+1)(2n+1) \frac{6}{6} \]
- \[ \sum_{k=1}^{n} k^3 = n^2(n+1)^2 \frac{4}{4} \]. This and the one above can be proved using induction.
- \[ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \], which is useful in sums involving many terms with this.

- \( x + \frac{1}{x} \geq 2 \) for real \( x \), which can be derived from the AM-GM inequality.

- Define a sequence \( a \) such that \( a_k = x^k + \frac{1}{x^k} \). If \( a_1 = z \), then \( a_{k+1} = za_k - a_{k-1} \). (We define \( a_{-1} = 2 \) here.)

Moreover, \( a_k = \frac{1}{2^k} \left( \left( z + \sqrt{z^2 - 4} \right)^k + \left( z - \sqrt{z^2 - 4} \right)^k \right) = \left[ \left( \frac{z + \sqrt{z^2 - 4}}{2} \right)^k + \frac{1}{2} \right] \).

The last formula only works when \( z > 2 \) and \( z \) is an integer. However, when \( z = 2 \), all terms of \( a \) will equal to 2.
References:

- Mathematics Database [http://www.mathdb.org/]
- MathsWorld2001 [http://mathsworld.ath.cx/]

Original documents:

- (None)